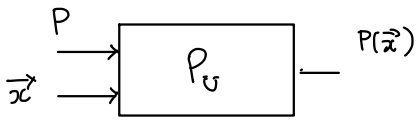
COMPUTABILITY (27/10/2026)



* Enumeration of URM programs

a set X is countable if $|X| \leq |\mathbb{N}|$

i.e. there is a surjection $f : \mathbb{N} \to X$ (enumeration)

$f(0)$ $f(1)$ $f(2)$ $f(3)$ ....

all elements of X (possibly with repetitions)

if f is injective it is called bijective enumeration

f is effective

Lemma : there are bijective enumerations (effective) of

(1) $\mathbb{N}^2$

(2) $\mathbb{N}^3$

(3) $\bigcup_{K \geq 1} \mathbb{N}^K$

proof

(1) $\pi : \mathbb{N}^2 \to \mathbb{N}$

$\pi(x,y) = 2^x (2y+1) \dot{-} 1$ computable

$\pi^{-1} : \mathbb{N} \to \mathbb{N}^2$ (effective)

$\pi^{-1}(m) = (\pi_1(m), \pi_2(m))$ $\pi_1, \pi_2 : \mathbb{N} \to \mathbb{N}$ computable

$m = \pi(x,y) = 2^x (2y+1) \dot{-} 1$

$m+1 = 2^x (2y+1)$

$\pi_1(m) = (m+1)_1$

$\pi_2(m) = qt(2, qt(2^{\pi_1(m)}, m+1) \dot{-} 1)$

$\pi(3,2) = 2^3 \cdot (2 \cdot 2 + 1) \dot{-} 1 = 39$

$\pi_1(39) = (39 + 1)$

$\pi_2(39) = qt(2, \underbrace{qt(2^3, 39+1)}_{5} \dot{-} 1) = 2$ (4)

(2) $\nu : \mathbb{N}^3 \to \mathbb{N}$

$\nu(x,y,z) = \pi(x, \pi(y,z))$ computable

$\nu^{-1} : \mathbb{N} \to \mathbb{N}^3$

$\nu^{-1}(m) = (\nu_1(m), \nu_2(m), \nu_3(m))$

$\nu_1(m) = \pi_1(m) \qquad \nu_2(m) = \pi_1(\pi_2(m)) \qquad \nu_3(m) = \pi_2(\pi_2(m))$

(3) $\tau : \bigcup_{k \geq 1} \mathbb{N}^k \to \mathbb{N}$

$\tau(x_1 \dots x_k) = \prod_{i=1}^{k} p_i^{x_i} \dot{-} 1$

not injective

$(1,0) \to p_1^1 \cdot \underbrace{p_2^0}_{1} \dot{-} 1 = 1$

$(1,0,0) \to p_1^1 \cdot \underbrace{p_2^0 \cdot p_3^0}_{1} \dot{-} 1 = 1$

$\tau(x_1, \dots, x_k) = \left(\prod_{i=1}^{k-1} p_i^{x_i} \right) p_k^{x_{k+1}} \dot{-} 2$

$\tau^{-1} : \mathbb{N} \to \bigcup_{k \geq 1} \mathbb{N}^k$

$a : \mathbb{N}^2 \to \mathbb{N}$

$\tau^{-1}(m) = (\underbrace{a(m,1) \quad a(m,2) \quad \dots \quad a(m, e(m))}_{e(m)})$

$e : \mathbb{N} \to \mathbb{N}$

$m = \left(\prod_{i=1}^{k-1} p_i^{x_i} \right) p_k^{x_{k+1}} \dot{-} 2$

$$m+2 = \left(\prod_{i=1}^{k-1} p_i^{x_i} \right) p_k^{x_{k+1}}$$

$\ell(m)$ = largest k ($\overset{m+2}{\swarrow}$) s.t. p_k divides $m+2$ — computable (exercise)

$= m \dot{-} \mu y < m+2 \,.\, \mathrm{div}(p_{m+2-y}, m+2)$

$$a(m,i) = \begin{cases} (m+2)_i & 1 \le i < \ell(m) \\ (m+2)_i \dot{-} 1 & i = \ell(m) \end{cases}$$

$\ell : \mathbb{N} \to \mathbb{N}$, $a : \mathbb{N}^2 \to \mathbb{N}$ computable functions

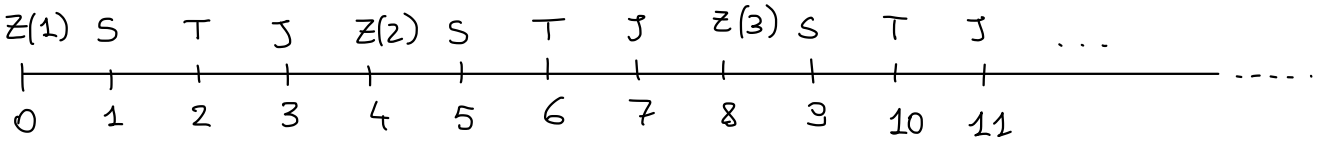
OBSERVATION : The class $\mathcal{P}$ of URM-programs admits an effective enumeration, bijective

$$\gamma : \mathcal{P} \to \mathbb{N}$$

Let $\mathcal{Y} = \{ Z(m), S(m), T(m,n), J(m,n,t) \mid m,n,t \in \mathbb{N},\ m,n,t \geq 1 \}$

we define

$$\beta : \mathcal{Y} \to \mathbb{N}$$



$\beta(Z(m)) = 4 * (m \dot{-} 1)$

$\beta(S(m)) = 4 * (m \dot{-} 1) + 1$

$\beta(T(m,n)) = 4 * \pi(m \dot{-} 1, n \dot{-} 1) + 2$

$\beta(J(m,n,t)) = 4 * \nu(m \dot{-} 1, n \dot{-} 1, t \dot{-} 1) + 3$

β invertible

$\beta^{-1} : \mathbb{N} \to \mathcal{Y}$ $\quad \beta(I) = x$ $\quad r = rm(4,x)$, $q = qt(4,x)$

$$\beta^{-1}(x) = \begin{cases} Z(q+1) & \text{if } r=0 \\ S(q+1) & \text{if } r=1 \\ T(\pi_1(q)+1,\ \pi_2(q)+1) & \text{if } r=2 \\ J(\nu_1(q)+1,\ \nu_2(q)+1,\ \nu_3(q)+1) & \text{if } r=3 \end{cases}$$

Given a program $P \in \mathcal{P}$ URM-program

$$P \begin{cases} I_1 \\ I_2 \\ \vdots \\ I_s \end{cases} \qquad \gamma(P) = \tau\left(\beta(I_1)\ \beta(I_2)\ \dots\ \beta(I_s)\right)$$

inverse

$$\gamma^{-1}: \mathbb{N} \to \mathcal{P}$$

$$\gamma^{-1}(m) = \begin{cases} I_1 \\ \vdots \\ I_{\ell(m)} \end{cases} \qquad I_i = \beta^{-1}(a(m,i))$$

* γ <u>fixed</u> enumeration of URM programs

$\gamma(P)$ (Gödel) number of P (code)

given m $P_m = \gamma^{-1}(m)$

<u>Examples</u>:

$\pi(0,1) = 2^0(2\cdot 1+1)-1 = 2$

* P $\begin{cases} T(1,2) \rightsquigarrow 4 * \pi(1-1, 2-1) + 2 = 10 \\ S(2) \rightsquigarrow 4 * (2-1) + 1 = 5 \\ T(2,1) \rightsquigarrow \text{- - - - - - - - -} = 6 \end{cases}$

$$\gamma(P) = \tau(10\ 5\ 6) = P_1^{10} \cdot P_2^{5} \cdot P_3^{6+1} \div 2$$

$$= 2^{10} \cdot 3^5 \cdot 5^7 \div 2$$

$$= 19\,439\,999\,998$$

* P^1 $\quad S(1)$

$$\gamma(P^1) = \tau(\beta(S(1))) = \tau(4*(1-1)+1) = \tau(1)$$

$$= p_1^{1+1} \dot{-} 2 = 2^2 \dot{-} 2 = 2$$

* given $\quad n = 40$

what is $\quad p_{40} = \gamma^{-1}(40)$ $\qquad \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) p_k^{x_{k+1}} \dot{-} 2$

$$42 = \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) p_k^{x_{k+1}}$$

$$= 2^1 \cdot 3^1 \cdot 7^1 = p_1^1 \cdot p_2^1 \cdot p_3^0 \cdot p_4^{\overbrace{0+1}^{1}}$$

$\ell(40) = 4$

$$\gamma^{-1}(40) = \begin{cases} I_1 & \beta^{-1}(1) & S(1) \\ I_2 & \beta^{-1}(1) & S(1) \\ I_3 & \beta^{-1}(0) & Z(1) \\ I_4 & \beta^{-1}(0) & Z(1) \end{cases}$$

* Fixed $\quad \gamma : \mathbb{P} \to \mathbb{N}$

this induces an enumeration of computable functions

$\varphi_m^{(k)} : \mathbb{N}^k \to \mathbb{N}$ $\qquad$ function of k arguments computed by program $\gamma^{-1}(m) = P_m$

$f_{P_m}^{(k)}$

$$W_m^{(k)} = \text{dom}(\varphi_m^{(k)}) = \{ \vec{x} \in \mathbb{N}^k \mid \varphi_m^{(k)}(\vec{x}) \downarrow \} \subseteq \mathbb{N}^k$$

$$E_m^{(k)} = \text{cod}(\varphi_m^{(k)}) = \{ \varphi_m^{(k)}(\vec{x}) \mid \vec{x} \in W_m^{(k)} \}$$

When $k=1$, we omit it

φ_m instead of $\varphi_m^{(1)}$

Example:

$$\varphi_{40} : \mathbb{N} \to \mathbb{N}$$

$$\varphi_{40}(x) = 0 \qquad \forall x \in \mathbb{N}$$

$$W_{40} = \mathbb{N}$$

$$E_{40} = \{0\}$$

Note

successor

successor

φ_0 φ_1 φ_2 φ_3 φ_4 ... $\varphi_{19\,439\,999\,998}$

enumeration of all unary computable function

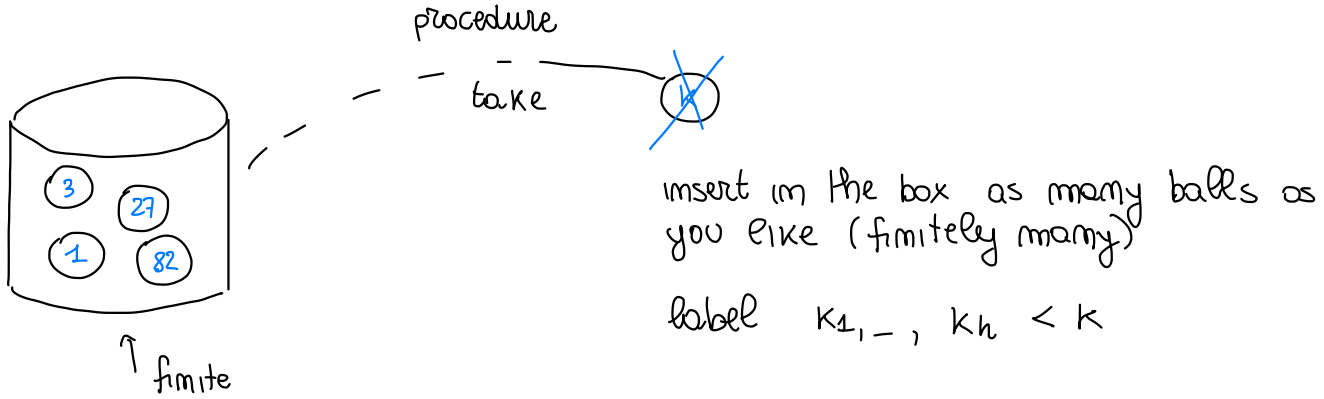
with repetitions (infinitely many)

$$|\mathcal{C}^{(1)}| \leq |\mathbb{N}|$$

$$|\mathcal{C}^{(k)}| \leq |\mathbb{N}|$$

$$\mathcal{C} = \bigcup_{k \geq 1} \mathcal{C}^{(k)} \qquad \text{is countable}$$

EXERCISE:



Is this process going to terminate?

EXERCISE :

partial recursive functions $\mathcal{R}$

least class of functions

- → including the BASIC FUNCTIONS
- → closed under
  1. COMPOSITION
  2. PRIMITIVE RECURSION
  3. UNBOUNDED MINIMALISATION

Originally defined by Gödel – Kleene $\mathcal{R}_0$

least class of functions

- → including the BASIC FUNCTIONS
- → closed under
  1. COMPOSITION
  2. PRIMITIVE RECURSION
  3. UNBOUNDED MINIMALISATION used only when the result is total

$$\mathcal{R}_0 \subseteq \mathcal{R} \cap \text{Tot}$$
$$\supseteq$$

Not obvious one can obtain total functions from partial ones

$$f : \mathbb{N}^2 \to \mathbb{N}$$

$$f(x,y) = \begin{cases} 1 & y < x \\ 0 & y = x \\ \uparrow & \text{otherwise} \end{cases} \quad \in \mathcal{R}$$

$$= sg(x \dot{-} y) + \mu z.\ y \dot{-} x$$

$$h(x) = \mu y.\ f(x,y) = x \qquad \text{total !}$$