COMPUTABILITY (26/10/2026)

* Class of partial recursive functions $\mathcal{R}$

least rich class of functions i.e. least class of functions

→ including the BASIC FUNCTIONS

→ closed under
1. COMPOSITION
2. PRIMITIVE RECURSION
3. UNBOUNDED MINIMALISATION

Theorem : $\mathcal{R} = \mathcal{C}$

proof

$(\mathcal{R} \subseteq \mathcal{C})$ $\mathcal{C}$ is rich

$(\mathcal{C} \subseteq \mathcal{R})$

Let $f : \mathbb{N}^k \to \mathbb{N}$ be a function in $\mathcal{C}$

and let P a URM-program for f

Define

$$\begin{cases} C_P^1 : \mathbb{N}^{k+1} \to \mathbb{N} \\ C_P^1(\vec{x}, t) = \text{content of register } R_1 \text{ after } t \text{ steps of } P(\vec{x}) \end{cases}$$

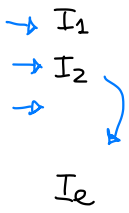
$$\begin{cases} J_P : \mathbb{N}^{k+1} \to \mathbb{N} \\ J_P(\vec{x}, t) = \begin{cases} \text{instruction to be executed after } t \text{ steps of } P(\vec{x}) \\ 0 \quad \text{if } P(\vec{x}) \text{ halts in } t \text{ or fewer steps} \end{cases} \end{cases}$$

Then

$$f(\vec{x}) = C_P^1(\vec{x}, \mu t.\, J_P(\vec{x}, t))$$

We conclude by proving $\boxed{C_P^1, J_P \in \mathcal{R}}$

program P for f (std form)



memory

r_1	r_2	r_3	- - - - - -	r_m	0	0	0

$$c = \prod_{i \geq 1} p_i^{r_i} = \prod_{i=1}^{m} p_i^{r_i}$$

$$r_i = (c)_i$$

1	2	3			
2	0	1	0	0	....

$$c = p_1^2 \cdot p_2^0 \cdot p_3^1 \cdot \overbrace{p_4^0 \cdot p_5^0}^{1}$$

$$= 2^2 \cdot 3^0 \cdot 5^1 = 20$$

$$(20)_3 = 1$$

$$\begin{cases} C_P : \mathbb{N}^{k+1} \to \mathbb{N} \\ C_P(\vec{x}, t) = \text{content of the memory after } t \text{ steps of } P(\vec{x}) \end{cases}$$

$$\begin{cases} J_P : \mathbb{N}^{k+1} \to \mathbb{N} \\ J_P(\vec{x}, t) = \begin{cases} \text{instruction to be executed after } t \text{ steps of } P(\vec{x}) \\ 0 \quad \text{if } P(\vec{x}) \text{ halts in } t \text{ or fewer steps} \end{cases} \end{cases}$$

we define C_P, J_P by primitive recursion

base case $C_P(\vec{x}, 0) = \prod_{i=1}^{k} p_i^{x_i}$

$J_P(\vec{x}, 0) = 1$

1		k		
x_1	- - -	x_k	0	- - -

recursion cases

we define $C_P(\vec{x}, t+1)$

$J_P(\vec{x}, t+1)$

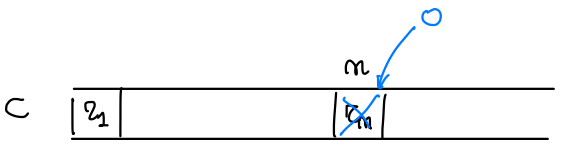
using

$C_P(\vec{x}, t) = C$

$J_P(\vec{x}, t) = J$

(notation)

$$c_p(\vec{x}, t+1) = \begin{cases} qt(p_m^{(c)_m}, c) & \text{if } 1 \le J \le \ell(P) \text{ and } I_J = z(m) \\ p_m \cdot c & \text{if } 1 \le J \le \ell(P) \text{ and } I_J = S(m) \\ p_m^{(c)_m} \cdot qt(p_m^{(c)_m}, c) & \text{if } 1 \le J \le \ell(P) \text{ and } I_J = T(m,m) \\ c & \text{otherwise } \left(\begin{array}{l} J = 0 \text{ or} \\ 1 \le J \le \ell(P) \text{ and} \\ \quad I_J = J(m,m,t) \end{array} \right) \end{cases}$$



$$c = p_1^{z_1} \cdot p_2^{z_2} \cdots p_m^{z_m} \cdot \cdots \qquad z_m = (c)_m$$

$$J_p(\vec{x}, t+1) = \begin{cases} J+1 & \text{if } 1 \le J < \ell(P) \text{ and } \left(\begin{array}{l} I_J = z(m), S(m), T(m,m) \\ \text{or } I_J = J(m,m,t) \\ \quad \text{with } (c)_m \neq (c)_m \end{array} \right) \\ u & \text{if } 1 \le J \le \ell(P) \text{ and } I_J = J(m,m,u) \text{ and } (c)_m = (c)_m \text{ and } 1 \le u \le \ell(P) \\ 0 & \text{otherwise} \end{cases}$$

Hence $c_p, J_p \in R$

and thus

$$f(x) = \left(c_p(\vec{x}, \mu t.\ J_p(\vec{x}, t)) \right)_1$$

therefore $f \in R$.

□

* Primitive Recursive Functions

PR = least class of functions

→ including the basic functions

→ closed under

① composition

② primitive recursion ~ for loop

③ ~~minimalisation~~ ~ while loop

$$PR \subsetneq ? \subsetneq \mathcal{R} \cap Tot$$

$$\| \qquad\qquad \|$$

$$\mathcal{C}_{for} \qquad \mathcal{C}_{for,while} \cap Tot$$

$$\wr \qquad\qquad \wr$$

$$\Psi \qquad\qquad \Psi$$

Ackermann Function

$\Psi : \mathbb{N}^2 \to \mathbb{N}$

$$\begin{cases} \Psi(0,y) = y+1 \\ \Psi(x+1,0) = \Psi(x,1) & (x+1,0) \geq_{lex} (x,1) \\ \Psi(x+1,y+1) = \Psi(x, \underbrace{\Psi(x+1,y)}_{u}) & (x+1,y+1) \geq_{lex} (x+1,y) \geq_{lex} (x,u) \end{cases}$$

We argue

① Ψ total

② $\Psi \in \mathcal{R} = \mathcal{C}$

③ $\Psi \notin PR$

$(\mathbb{N}^2, \leq_{lex})$ $\quad (x,y) \leq_{lex} (x',y')$ if $(x < x')$ or $(x = x'$ and $y \leq y')$

ex: $(1, 1000) <_{lex} (2,0)$

$(1, 1000) <_{lex} (1, 1001)$

recurring on smaller parameters is not sufficient

$f: \mathbb{Z} \to \mathbb{Z}$

$$f(z) = \begin{cases} 0 & z \geq 0 \\ f(z-1) & z < 0 \end{cases}$$

$f(-1) = ?$

$f(-1)$
$=$
$f(-2)$
$=$
$f(-3)$
$\vdots$

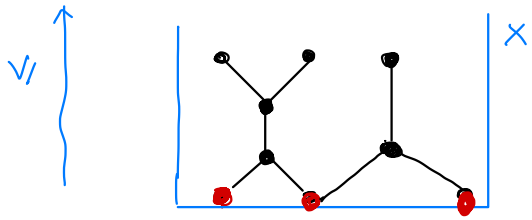
① ψ is total

* partially ordered set

$(D, \leq)$ $\leq$ reflexive $x \leq x$
antisymmetric $x \leq y \;\&\; y \leq x \Rightarrow x = y$
transitive $x \leq y \;\&\; y \leq z \Rightarrow x \leq z$

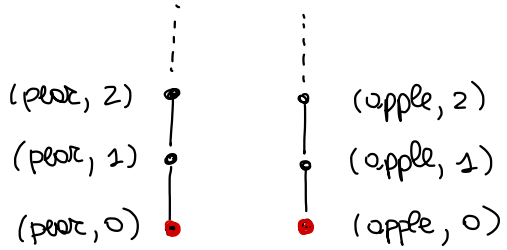
* Well founded partial order

$(D, \leq)$ well founded if $\forall X \subseteq D$ $X \neq \emptyset$ X includes a minimal element



$d \in X$ minimal if $d' \leq d$ then $d' = d$

$D = \{ (pear, m), (apple, m) \mid m \in \mathbb{N} \}$



$(t, m) \leq (t', m)$ if $t = t'$ and $m \leq m$

$\mathbb{Z}$ not well founded

$\mathbb{N}$ is well founded

NOTE : $(D, \leq)$ is well-founded

iff

there are not infinite descending chains in D

$d_0 > d_1 > d_2 > \dots$

* $(\mathbb{N}^2, \leq_{ex})$ well founded

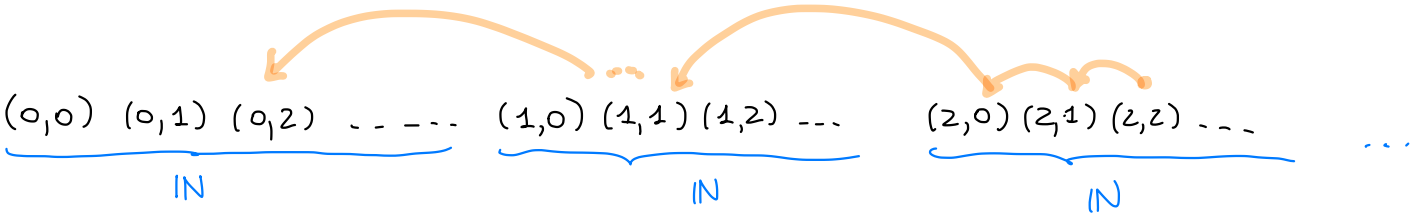
let $X \subseteq \mathbb{N}^2$ $X \neq \emptyset$

$x_0 = \min \{ x : \exists y. (x,y) \in X \}$

$y_0 = \min \{ y \mid (x_0, y) \in X \}$

then $(x_0, y_0) = \min X$ (in $\leq_{ex}$)

↳ ψ is total



More precisely one should resort to a corresponding induction principle

[NOT DISCUSSED IN THE CLASS

Induction : $P(n)$ $\forall n \in \mathbb{N}$ $P(n) = "\sum_{i=0}^{n} i = \frac{n(n+1)}{2}"$

$P(0)$ and for all $n \in \mathbb{N}$ if $P(n)$ then $P(n+1)$

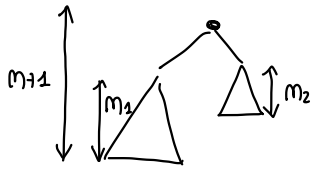
$\Downarrow$

for all $n \in \mathbb{N}$ $P(n)$

Strong Induction

A binary tree with height n has at most 2^n leaves

$n \to n+1$



$n_1, n_2 \leq n$

need inductive hypothesis for all $n' \leq n$

for all n, assuming $P(n')$ for $n' < n$ I can deduce $P(n)$

$\Downarrow$

for all $n \in \mathbb{N}$ $P(n)$

Well-founded induction

$(D, \leq)$ well founded order

$P(x)$ property of elements of D

if for all $d \in D$, assuming $\forall d' < d$ $P(d')$ I can conclude $P(d)$

$\Downarrow$

$\forall d \in D$ $P(d)$

ψ is total

$\forall (x,y) \in \mathbb{N}$ $\psi(x,y)\downarrow$

by well-founded induction on $(\mathbb{N}, \leq_{lex})$

proof

let $(x,y) \in \mathbb{N}^2$

assume $\forall (x',y') <_{lex} (x,y)$ $\psi(x',y')\downarrow$

and we show $\psi(x,y)$

we distinguish 3 cases

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1,0) = \psi(x,1) \\ \psi(x+1,y+1) = \psi(x,\psi(x+1,y)) \end{cases}$$

$(x=0)$ $\quad \psi(x,y) = \psi(0,y) = y+1 \downarrow$

$(x>0,\ y=0)$ $\quad \psi(x,y) = \psi(\underbrace{x-1,1}) \downarrow$

$<_{lex}(x,y)$ hence $\psi(x-1,1)\downarrow$

$(x>0,\ y>0)$ $\quad \psi(x,y) = \psi(x-1, \psi(\underbrace{x,y-1})) = \psi(\underbrace{x-1,u}) \downarrow$

$<_{lex}(x,y)$ $\quad <_{lex}(x,y)$

$\rightsquigarrow \psi(x,y-1) = u \downarrow$ $\quad \rightsquigarrow \psi(x-1,u)\downarrow$

□

② $\psi \in \mathcal{R} = \mathcal{C}$

$\psi(1,1)$

$=$

$\psi(0, \underbrace{\psi(1,0)}) = 3$

$=$

$\psi(0,1)$

$=$

2

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1,0) = \psi(x,1) \\ \psi(x+1,y+1) = \psi(x,\psi(x+1,y)) \end{cases}$$

$(1,1,3)$ $\quad (0,2,3)$ $\quad (1,0,2)$ $\quad (0,1,2)$

valid set of tuples : a set $S \subseteq \mathbb{N}^3$ such that

$(x,y,z) \in \mathbb{N}^3 \rightarrow z = \psi(x,y)$

$\rightarrow$ the set contains all triples needed to compute $\psi(x,y)$

more formally : $S \subseteq \mathbb{N}^3$ is <u>valid</u>

(1) $(0,y,z) \in S \Rightarrow z = y+1$

(2) $(x+1,0,z) \in S \Rightarrow (x,1,z) \in S$

(3) $(x+1,y+1,z) \in S \Rightarrow$

$\exists u \quad (x+1,y,u) \in S$ and $(x,u,z) \in S$

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1,0) = \psi(x,1) \\ \psi(x+1,y+1) = \psi(x,\psi(x+1,y)) \end{cases}$$

One can prove that $\forall (x,y,z) \in \mathbb{N}^3$

$\psi(x,y) = z$ iff there is a <u>finite valid</u> set of triples $S \subseteq \mathbb{N}^3$ s.t. $(x,y,z) \in S$

then

$$\psi(x,y) = \text{"}\ \mu\ \underbrace{(S,z)}_{\text{encode as a number}} \quad \left(\begin{array}{l} S \subseteq \mathbb{N}^3 \text{ is finite valid set of triples} \\ \& \quad (x,y,z) \in S \end{array} \right) \text{"}$$

$$S = \{(x_1,y_1,z_1), (x_2,y_2,z_2), \ldots, (x_m,y_m,z_m)\}$$

$$\{\pi(\pi(x_1,y_2),z_1), \quad \text{---} \quad \pi(\pi(x_m,y_m),z_m)\}$$

k_1 ---- k_m

$$\prod_{i=1}^{m} p_i^{k_i}$$

$\rightsquigarrow \quad \psi \in \mathcal{R} = \mathcal{C}$

③ $\psi \notin PR$

successor

$$\begin{cases} x+0 = x \\ x+(y+1) = (x+y)+1 \end{cases}$$ ← successor iterated y times

$$\begin{cases} x*0 = 0 \\ x*(y+1) = (x*y)+x \end{cases}$$ ← "$+x$" iterated y times

$$\begin{cases} x^0 = 1 \\ x^{y+1} = x^y * x \end{cases}$$ ← "$*x$" iterated y times

⋮

idea: ψ brings the above to the "limit"

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1,0) = \psi(x,1) \\ \psi(x+1,y+1) = \psi(x,\psi(x+1,y)) \end{cases}$$

$$\begin{cases} \psi_0(y) = y+1 \\ \psi_{x+1}(0) = \psi_x(1) \\ \psi_{x+1}(y+1) = \psi_x(\psi_{x+1}(y)) \end{cases}$$

consider x as fixed parameter

$$\psi_x(y) = \psi(x,y)$$

$$\begin{aligned} \psi_{x+1}(y+1) &= \psi_x(\psi_{x+1}(y)) \\ &= \psi_x\,\psi_x\,(\psi_{x+1}(y-1)) \\ &= \psi_x\,\psi_x\,\psi_x\,(\psi_{x+1}(y-2))\ldots \\ &\vdots \\ &= \underbrace{\psi_x \text{ --- } \psi_x}_{y+1 \text{ times}}\ \underbrace{\psi_{x+1}(0)}_{\psi_x(1)} \\ &= \psi_x^{y+2}(0) \end{aligned}$$

roughly: increasing x to $x+1$ requires iterating the function ψ_x

→ increases the number of nested primitive recursion

→ the full function would require infinitely many nested primitive recursions

Some more ideas.... [not discussed in the class]

concretely:

$\psi_0(y) = y+1$

$\psi_1(y) = \psi_0^{y+1}(1) = y+2$

$\psi_2(y) = \psi_1^{y+1}(1) = 2(y+1)+1 = 2y+3 \approx 2y$

$\psi_3(y) = \psi_2^{y+1}(1) \simeq 2^y$

$\psi_4(y) = \psi_3^{y+1}(1) \simeq 2^{2^{2^{2^{y}}}}$

eg: $\psi_0(1) = 2$

$\psi_2(1) = 5$

$\psi_3(1) = 13$

$\psi_4(1) \simeq 2^{16}$

$\psi_4(2) \simeq 2^{2^{16}} \simeq 10^{6400}$

ONE CAN PROVE: Given a function $f: \mathbb{N}^m \to \mathbb{N} \in PR$ and a program P computing f using only "for-loops" (primitive recursion) if J is the maximum level of nesting of for-loops

$$f(\vec{x}) < \psi_{J+1}(\max\{x_i\})$$

Now, assume $\psi \in PR$, let J be the level of nesting of for-loops (of primitive recursive defs) for computing ψ

$\forall (x,y)$

$$\psi(x,y) < \psi_{J+1}(\max\{x,y\})$$

Let $x = y = J+1$

$$\Psi(J+1, J+1) < \Psi_{J+1}(J+1) = \Psi(J+1, J+1)$$

contradiction

$\Rightarrow \quad \Psi \notin \mathcal{PR}$