Another example of a distribution of order 1.

$\in \mathcal{D}'(\mathbb{R})$ · PRINCIPAL VALUE p.v $\frac{1}{x}$

$$T(\varphi) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,dx = \text{p.v} \int_{\mathbb{R}} \frac{\phi(x)}{x}\,dx \qquad \forall\, \phi \in C_c^\infty(\mathbb{R})$$

↓ ASSOCIATED WITH $f(x) = \frac{1}{x} \notin L^1_{loc}(\mathbb{R})$!

well defined fix $\phi \in C_c^\infty(\mathbb{R})$ $\exists\, a > 0$ such that $\operatorname{supp}\phi \subset (-a,a)$

$$\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,dx = \int_{-a}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{a} \frac{\phi(x)}{x}\,dx = \int_{-a}^{-\varepsilon} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{-a}^{-\varepsilon} \frac{\phi(0)}{x}\,dx$$

$$+ \int_{\varepsilon}^{a} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{\varepsilon}^{a} \frac{\phi(0)}{x}\,dx \qquad \frac{\phi(0)}{x} \text{ is ODD !}$$

$$\frac{\phi(x)-\phi(0)}{x} = \int_0^x \frac{\phi'(y)}{x}\,dy = \int_0^1 \frac{\phi'(tx)\,x}{x}\,dt = \int_0^1 \phi'(tx)\,dt$$

$y = xt$

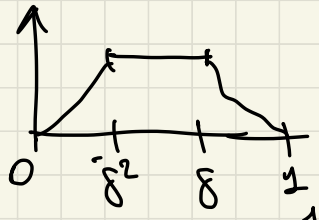
$$\lim_{\varepsilon\to 0^+} \int_{-a}^{-\varepsilon}\int_0^1 \phi'(tx)\,dt\,dx + \int_{\varepsilon}^{a}\int_0^1 \phi'(tx)\,dt\,dx =$$

$$= \int_{-a}^{a}\int_0^1 \phi'(tx)\,dt\,dx = \text{p.v.}\,\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)}{x}\,dx$$

① $\left|\text{p.v.}\,\frac{1}{x}(\phi)\right| = \left|\int_{-a}^{a}\int_0^1 \phi'(tx)\,dt\,dx\right| \le \|\phi'\|_\infty \,(\text{diam } k)$ | ORDER ≤ 1

if $\text{supp}\,\phi \subseteq k$.

② Note that it cannot have order 0

take ϕ_δ 

$\delta > 0$ $\quad \phi_\delta \equiv 1$ on (δ^2, δ) $\quad \varphi \ge 0\ \forall x$

$\phi_\delta \equiv 0 \quad x \le 0 \quad x \ge 1$

$\phi \in C_c^\infty(\mathbb{R})$

$$\text{p.v.}\,\frac{1}{x}(\phi_\delta) = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{\phi_\delta(x)}{x} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\delta} \frac{\phi_\delta(x)}{x} + \int_{\delta^2}^{\delta} \frac{\phi_\delta(x)}{x} + \int_\delta^1 \frac{\phi_\delta(x)}{x}\,dx$$

$$\ge \int_{\delta^2}^{\delta} \frac{1}{x}\,dx = \log\delta - \log\delta^2 = -\log\delta$$

Therefore order $(p.v.\frac{1}{x}) = 1$

(Note that $p.v.\frac{1}{x} \notin L^1_{loc}(\mathbb{R})$)

possible to extend to dim n.

Ex Let $T: C_c^\infty(U) \to \mathbb{R}$ positive distribution

$(\forall \phi \geq 0 \; T(\phi) \geq 0)$

Show that T has order 0.

(Some idea that proving T linear positive functional on $C_c(U)$ is continuous...)

Def SUPPORT of DISTRIBUTION: $\operatorname{supp} T = U \setminus \omega_0$ (closed in U)

where $\omega_0 = \bigcup \{ A \subseteq U \text{ open s.t. } \forall \varphi \in \mathcal{C}_c^\infty(U) \ \operatorname{supp}\varphi \subseteq A \ \ T\varphi = 0 \}$.

So if $\phi \equiv 0$ on some A open set with $\operatorname{supp} T \subseteq A \Rightarrow T\varphi = 0$

If $\phi = 0$ on $\operatorname{supp} T \not\Rightarrow T\varphi = 0$ (ex: $T(\varphi) = \varphi'(0)$ in $\mathcal{D}'(\mathbb{R})$ $\operatorname{supp} T = \{0\}$, $\varphi(0) = 0 \not\Rightarrow \varphi'(0) = 0$!)

Prop If $\operatorname{supp} T$ is COMPACT then T has finite order

proof $K = \operatorname{supp} T$ (compact)

Let $K \subset\subset U_1 \subset\subset U_2 \subset\subset U$ $\quad U_1, U_2$ bdd open in U

Take $\psi \in \mathcal{C}_c^\infty(U)$ such that $\psi \equiv 1$ on $\overline{U}_1$ $\operatorname{supp}\psi \subseteq \overline{U}_2$

$\Rightarrow \forall \varphi \in \mathcal{C}_c^\infty(U) \quad \varphi\cdot\psi \in \mathcal{C}_c^\infty(U)$ $\operatorname{supp}\varphi\psi \subseteq \overline{U}_2$

$\varphi(1-\psi) = 0$ in $U_1 \Rightarrow T(\varphi(1-\psi)) = 0 \Rightarrow T\varphi = T(\varphi\psi)$

$$|T\varphi| = |T(\varphi\psi)| \leq C_{K_2} \sup_{|\alpha| \leq p_{K_2}} \|D^\alpha \varphi\psi\|_\infty \leq \tilde{C}_{K_2}(\psi) \sup_{|\alpha| \leq p_{K_2}} \|D^\alpha \varphi\|_\infty$$

ORDER $\leq p_{K_2}$

CONVERGENCE IN THE SENSE OF DISTRIBUTIONS.

$T_n \to T$ in $\mathcal{D}'(U)$ means $T_n(\varphi) \to T(\varphi)$ $\forall \varphi \in \mathcal{C}_c^\infty(U)$

Ex $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ $\qquad \rho(x) = \begin{cases} C e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$

$\rho_\varepsilon \to \delta_0$ in the sense of distributions in $\mathcal{D}'(\mathbb{R}^n)$

NB $\rho_\varepsilon \to 0$ a.e.! $\|\rho_\varepsilon\|_{L^1} = 1$ $\forall \varepsilon$

Thesis is $T_{\rho_\varepsilon}(\varphi) = \int_{\mathbb{R}^n} \varphi \rho_\varepsilon(x)\,dx \overset{y = x/\varepsilon}{=} \int_{B(0,1)} \varphi(\varepsilon y)\,\rho(y)\,dy \to \varphi(0)$

$f_n \to f$ in the sense of distributions if $\int_U f_n \phi\,dx \to \int_U f \phi\,dx$ $\forall \phi \in \mathcal{C}_c^\infty(U)$

Ex f_n bold in L^p $f_n \to f$ in the sense of distributions
is it true that $f_n \to f$ in L^p? $p=1$ $f_n = \chi_{[n,n+1]}(x)$

CONVOLUTION

$T \in \mathcal{D}'(U)$ $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ $V_\varphi = \{ x \text{ s.t. } \forall y \in \operatorname{supp}\varphi \quad x-y \in U \}$ ($x - \operatorname{supp}\varphi \in U$)

$\forall x \in V_\varphi \quad T * \varphi(x) := T(\varphi^x)$ $\qquad \varphi^x(y) = \varphi(x-y) \in \mathcal{C}^\infty_c(U)$.

Note that if $T = T_f$ $\quad T_f * \varphi(x) = T_f(\varphi^x) = \int_U f(y)\varphi(x-y)\,dy = f * \varphi(x)$.

① $T * \varphi \in \mathcal{C}^\infty(V_\varphi)$ since $D^\alpha(T*\varphi)(x) = T * D^\alpha\varphi(x)$

$$\frac{\partial}{\partial x_1}[T*\varphi(x)] = \lim_{h\to 0} T\left(\frac{\varphi^{x+he_1} - \varphi^x}{h}\right) \to T\left(\frac{\partial}{\partial x_1}\varphi(x-y)\right) = T\left(\frac{\partial}{\partial x_1}\varphi^x\right)$$

$x \in V_\varphi$ $\quad x + he_1 \in V_\varphi$ for h small

$\psi_h(y) := \dfrac{\varphi(x+he_1) - \varphi(x-y)}{h} \xrightarrow{h\to 0} \dfrac{\partial}{\partial x_1}\varphi(x-y)$ in $\mathcal{C}^\infty_c(U)$

② δ_0 is the unit of convolution $\quad \delta_0(\varphi) = \varphi(0)$

$$\delta_0 * \varphi(x) = \delta_0(\varphi^x) = \varphi^x(0) = \varphi(x)$$

$$\delta_0 * \varphi \equiv \varphi$$

(3) Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\phi(x) = \phi(-x)$

Let $T \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ $\quad V_\phi = \mathbb{R}^n$

$\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ $\quad T(\varphi * \psi) = \int_{\mathbb{R}^n} (T * \varphi)(x)\, \psi(x)\, dx$

$= \int_{\mathbb{R}^n} T(\varphi^x)\, \psi(x)\, dx$

Observe that $\phi * \psi(y) = \int_{\mathbb{R}^n} \phi(y-x)\, \psi(x)\, dx =$

$=$ by symmetry of ϕ $= \int_{\mathbb{R}^n} \phi(x-y)\, \psi(x)\, dx =$

$= \int_{\mathbb{R}^n} \phi^x(y)\, \psi(x)\, dx$ $\qquad y \to \phi * \psi(y) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$T(\phi * \psi) = T\left[\int_{\mathbb{R}^n} \phi^x(y)\, \psi(x)\, dx\right] =$ by linearity $= \int_{\mathbb{R}^n} T(\phi^x)\, \psi(x)\, dx$

Theorem (density of test functions in the space of distributions).

Let $T \in \mathcal{D}'(U)$. Then the following holds:

(1) $T * \rho_\varepsilon \to T$ in the sense of distributions in $\mathcal{D}'(A)$ $\forall A \subsetneq U$ open set.

(2) If T has compact support $T * \rho_\varepsilon \to T$ in $\mathcal{D}'(U)$.

(3) $\exists\, \phi_\varepsilon \in C_c^\infty(U)$ s.t. that $\phi_\varepsilon \xrightarrow{\mathcal{E}} T$ in the sense of distributions (that is $T_{\phi_\varepsilon} \to T$ that is $\forall \psi \in C_c^\infty(U)$ $\lim_{\varepsilon \to 0} \int_U \psi \phi_\varepsilon \, dx \to T(\psi)$).

Obs the important result is (3).

Lemma Let $\phi \in C_c^\infty(U)$ Then $\phi * \rho_\varepsilon \to \phi$ in $C_c^\infty(U)$.

proof Let ε_0 such that $\operatorname{supp}(\phi * \rho_{\varepsilon_0}) \subseteq \operatorname{supp}\phi + B(0,\varepsilon_0) \subset\subset U$

for $\varepsilon \le \varepsilon_0$ $\operatorname{supp}(\phi * \rho_\varepsilon) \subseteq \operatorname{supp}(\phi * \rho_{\varepsilon_0}) = K_0 \subset\subset U$

Moreover $\forall \alpha \in \mathbb{N}^n$ $D^\alpha(\phi * \rho_\varepsilon) = D^\alpha \phi * \rho_\varepsilon \to D^\alpha \phi$ uniformly. □

proof of the thm

① + ② $V_\varepsilon = \{x \quad x + \operatorname{supp}\rho_\varepsilon \subseteq U\} = \{x \in U \quad d(x, \mathbb{R}^n \setminus U) > \varepsilon\}$

$T * \rho_\varepsilon \in C^\infty(V_\varepsilon)$ Note that $V_{\varepsilon_1} \subseteq V_{\varepsilon_2}$ $\varepsilon_1 \ge \varepsilon_2$

$\bigcup_{\varepsilon > 0} V_\varepsilon = U$

Let $A \subsetneq U$ open $\Rightarrow \exists \varepsilon_0$ $A \subseteq V_{\varepsilon_0} \Rightarrow T * \rho_\varepsilon \in C^\infty(A)$ $\forall \varepsilon \le \varepsilon_0$

If $\operatorname{supp} T$ is compact, $\operatorname{supp} T \subseteq V_{\varepsilon_0}$ $\exists \varepsilon_0 \Rightarrow T * \rho_\varepsilon \in C^\infty(U)$ $\forall \varepsilon \le \varepsilon_0$

Let $\psi \in \mathcal{C}_c^\infty(A) \to \exists \varepsilon_0$ such that $\operatorname{supp}\psi \subseteq V_{\varepsilon_0}$

$\varepsilon \leq \varepsilon_0$ Recall $A \subseteq V_\varepsilon$ property of convolution n°3

$$T_{T*\varrho_\varepsilon}(\psi) = \int_A \psi(y)\, T*\varrho_\varepsilon(y)\,dy = T(\varrho_\varepsilon * \psi)$$

$\varrho_\varepsilon * \psi \to \psi$ in $\mathcal{C}_c^\infty(A)$ by Leeeeee

$T(\varrho_\varepsilon * \psi) \to T(\psi)$ by def of distribution.

$$T_{T*\varrho_\varepsilon}(\psi) \to T(\psi) \quad \forall \psi \in \mathcal{C}_c^\infty(A) \Rightarrow T*\varrho_\varepsilon \to T \text{ in the sense of distribut.}$$

Then $T*\varrho_\varepsilon \to T$ in $\mathcal{D}'(A)$ $\forall A$ open set of U

$A \subsetneq U$.

If T has compact support, we get $T*\varrho_\varepsilon \to T$ in $\mathcal{D}'(U)$.

③ We reduce to T of compact support.

$U_\varepsilon = \{ x \in V_\varepsilon \quad |x| < \frac{1}{\varepsilon} \} \subseteq V_\varepsilon \qquad U_\varepsilon \subset\subset U$

$U_{\varepsilon_1} \subseteq U_{\varepsilon_2}$ if $\varepsilon_1 \geq \varepsilon_2$

Let $\xi_\varepsilon \in C_c^\infty(U)$ with $\xi_\varepsilon \equiv 1$ on U_ε $\xi_\varepsilon \equiv 0$ $U \setminus U_{\varepsilon/2}$

Let us define $T_\varepsilon(\varphi) = T(\varphi \xi_\varepsilon) \quad \forall \varphi \in C_c^\infty(U)$.

$T_\varepsilon \in \mathcal{D}'(U)$, $\operatorname{supp} T_\varepsilon \subseteq \overline{U_{\varepsilon/2}}$, $T_\varepsilon \longrightarrow T$ in $\mathcal{D}'(U)$

since $\forall \phi \in C_c^\infty(U) \ \exists \varepsilon_0$ s.t. that $\phi \xi_\varepsilon \equiv \phi \quad \forall \varepsilon \leq \varepsilon_0 \Rightarrow T_\varepsilon(\phi) = T(\phi)$.

Fix $\varepsilon_0 > 0 \Rightarrow T_{\varepsilon_0} * \rho_\delta \longrightarrow T_{\varepsilon_0}$ in $\mathcal{D}'(U)$ by (1)

then pass to the limit as $\varepsilon_0 \to 0$. $(T_{\varepsilon_0} \to T)$ $\square$.