AREA FORMULA (by Federer '50)

↑ generalize the change of variable formula

$f : \mathbb{R}^k \to \mathbb{R}^m \quad k \leq m$ of class C^1 (same result for f Lipschitz)

$A \subseteq \mathbb{R}^k$ measurable

$$\int_A J_k f(x)\, dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^k(y)$$

where $J_k f(x) = \sqrt{\det(Df(x)^T Df(x))}$

→ If f is injective $\int_A J_k f(x)\, dx = \mathcal{H}^k(f(A))$

Case $f = T$ linear $\quad T : \mathbb{R}^k \to \mathbb{R}^m$ linear $k \le m$

$x \mapsto A \cdot x$ $\quad A$ matrix

Then $\mathcal{H}^k(T(E)) = J_k(T)\, \mathcal{H}^k(E) = J(T)\,|E| \quad \forall E \in \mathcal{B}(\mathbb{R}^k)$

$J_k(T) = \sqrt{\det(A^t A)}$

How to prove? $T : \mathbb{R}^k \to \mathbb{R}^m$ Let $R : \mathbb{R}^m \to \mathbb{R}^m$ rotation such that $\operatorname{Im}(R \circ T) \subseteq \{x_{k+1} = 0 = x_{k+2} = \dots = x_m\}$

$$\mathbb{R}^k \xrightarrow{T} \mathbb{R}^m \xrightarrow{R} \mathbb{R}^m \xrightarrow{\pi_k} \mathbb{R}^k \qquad \pi_k(x) = (x_1, \dots x_k)$$

$|\pi_k(R \circ T(E))| = J(T)\,|E|$ change of variable in $\mathbb{R}^k$

$\mathcal{H}^k(\pi_k(R \circ T(E))) = \mathcal{H}^k(R \circ T(E)) = \mathcal{H}^k(T(E))$

the general case much more complicated..

C^1 k-dimensional manifold $\quad k < n$

M is a k-dim. manifold in $\mathbb{R}^n$ if

$\forall x \in M \ \exists U_x \subseteq M$, V open in $\mathbb{R}^k$ $f \in C^1$ $f: V \to U_x$

which is injective and Df injective $\forall y \in V$

f is a chart or a parametrization

Moreover M is covered by a countable number of charts.

$\forall A \subseteq f(V) \subseteq M \qquad \mathcal{H}^k(A) = \int_{f^{-1}(A)} Jf \, dx$

if $g \geq 0 \quad g: \mathbb{R}^n \to \mathbb{R} \qquad \int_A g(x) \, d\mathcal{H}^k(x) = \int_{f^{-1}(A)} g(f^{-1}(y)) \, Jf(y) \, dy$

$\mathcal{H}^k \llcorner_M$ is the surface measure (= "dS")

Ex 1 $f: \mathbb{R} \to \mathbb{R}^m$ C^1 (CURVE)

LENGHT of a CURVE

$Df = (f_1', \dots f_m')$ $Jf = \sqrt{(f_1')^2 + \dots + (f_m')^2}$

$C = f[a,b]$ curve

f injective in $[a,b)$

$$\mathcal{H}^1(C) = \int_a^b Jf(x)\,dx$$

Ex 2 AREA OF THE GRAPH

$g: \mathbb{R}^n \to \mathbb{R}$ $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$, $x \mapsto (x, g(x))$

$$Df = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \frac{\partial g}{\partial x_1} & \dots & \frac{\partial g}{\partial x_m} \end{pmatrix}$$

$$Jf = \sqrt{1 + |Dg|^2} \qquad \mathcal{H}^n\{(x, g(x)),\ x \in U\} = \int_U \sqrt{1 + |Dg|^2}\,dx$$

M is a K-dim. C^1 manifold $\Rightarrow \mathcal{H}^k_{\lfloor M}$ is the surface measure (used to compute integrals on M).

We may extend integration on a larger sets of dimension $k < n$ called RECTIFIABLE sets

(useful since we need to "close" the family of C^1 k-dimensional manifolds)

def $E \subseteq \mathbb{R}^n$ is COUNTABLY k-RECTIFIABLE if

$\exists (f_i)_{i \in \mathbb{N}}$ at most countable $f_i : \mathbb{R}^k \to \mathbb{R}^n$ C^1

such that $\mathcal{H}^k(E \setminus \bigcup_i f_i(\mathbb{R}^k)) = 0$

E is k-RECTIFIABLE if it is countably k-rect. and $\mathcal{H}^k(E) < +\infty$.

So E can be covered up to a set of $\mathcal{H}^k$ measure 0 by C^1 k-dimensional manifold.

Def $E \subseteq \mathbb{R}^n$ is PURELY k-UNRECTIFIABLE if

$\forall f : \mathbb{R}^k \to \mathbb{R}^n$ f of class C^1 $\mathcal{H}^k(E \cap f(\mathbb{R}^k)) = 0$

(in particular $\mathcal{H}^k(E \cap M) = 0$ $\forall M$ k-dim. C^1 manifold)

Typical example CANTOR DUST

Ex $E_1, E_2 \subseteq \mathbb{R}$ $|E_1| = 0 = |E_2|$ and $\dim_{\mathcal{H}} E_1 = 1 = \dim_{\mathcal{H}} E_2$

then $E_1 \times E_2$ is purely 1-unrectifiable in $\mathbb{R}^2$.

Let γ be a curve in $\mathbb{R}^2$ $\{(t, f(t))\} \cap (E_1 \times E_2) \subseteq \{(t, f(t)),\ t \in E_1\}$

$\mathcal{H}^1((t, f(t)) \cap E_1 \times E_2) \leq \mathcal{H}^1(f(E_1)) \leq \sqrt{1 + \|f'\|_\infty}\, \mathcal{H}^1(E_1) = 0$ ($f(E_1) \sim E_1$)

Besikovitch $E \subseteq \mathbb{R}^2$ is purely 1-unrectifiable iff $\mathcal{H}^1(\pi(E)) = 0$ $\forall \pi$ project.
$\mathcal{H}^1(E) > 0$

DISTRIBUTION THEORY

REFERENCE: CHAPTER 9 FOLLAND

(generalization of the concept of functions and measures
L. Schwartz, '50)

$U \subseteq \mathbb{R}^n$ open

$C_c^\infty(U)$ test functions (with the topology that we put
$(= \mathcal{D}(U))$ ↳ INDUCTIVE LIMIT of FRECHET SPACES

MONTEL SPACE: CLOSED BDD SETS ARE COMPACTS

Def: A LINEAR functional $T: C_c^\infty(U) \longrightarrow \mathbb{R}$ is a DISTRIBUTION if it is (SEQUENTIALLY) CONTINUOUS w.r.t. the convergence defined in $C_c^\infty(U)$

that is $\phi_m \longrightarrow \phi$ in $C_c^\infty(U) \Rightarrow T(\phi_m) \longrightarrow T(\phi)$ (in $\mathbb{R}$)

REMEMBER $\exists K \subset\subset U$
$\operatorname{supp} \phi_m, \operatorname{supp} \phi \subseteq K \quad \|D^\alpha \phi_m - D^\alpha \phi\|_\infty \to 0 \ \forall \alpha$

EQUIVALENT DEFINITION $T: \mathcal{C}_c^\infty(U) \to \mathbb{R}$ linear

It is equivalent ① T sequentially continuous

and ② $\forall K \subset\subset U$ $\exists\, C_K > 0$, $\exists\, p_K \in \mathbb{N}$

such that $\forall \phi \in \mathcal{C}_c^\infty(U)$ $\operatorname{supp}\phi \subseteq K$

it holds $|T(\varphi)| \le C_K \sup_{|\alpha| \le p_K} \|D^\alpha \phi\|_\infty$

② $\Rightarrow$ ① Take $f_n \to f$ in $\mathcal{C}_c^\infty(U) \Rightarrow \exists K \subset\subset U$ such that $\operatorname{supp} f_n, \operatorname{supp} f \subseteq K \subset\subset U$. $\|D^\alpha(f_n - f)\|_\infty \to 0 \ \forall \alpha$

by ② applied to $f_n - f$: $|T(f_n - f)| \le C_K \sup_{|\alpha| \le p_K} \|D^\alpha(f_n - f)\|_\infty \to 0$

$\Rightarrow |T(f_n - f)| \to 0 \Rightarrow Tf_n \to Tf$

① ⇒ ② By contradiction: assume ② does not hold and ① holds.

So $\exists K \subset\subset U$ $\forall j \in \mathbb{N}$ $\exists \phi_j \in C_c^\infty(U)$ with $\operatorname{supp} \phi_j \subseteq K$

$$|T(\phi_j)| \geq j \cdot \max_{|\alpha| \leq j} \|D^\alpha \phi_j\|_\infty$$

Set $\psi_j = \dfrac{\phi_j}{|T(\phi_j)|} \in C_c^\infty(U)$ $\operatorname{supp} \psi_j \subseteq K$, $|T\psi_j| = 1$ by linearity

$$j \cdot \sup_{|\alpha| \leq j} \|D^\alpha \psi_j\|_\infty = j \sup_{|\alpha| \leq j} \frac{\|D^\alpha \phi_j\|_\infty}{|T(\phi_j)|} \leq 1$$

$\sup_{|\alpha| \leq j} \|D^\alpha \psi_j\|_\infty \leq \frac{1}{j} \Rightarrow \forall m \in \mathbb{N}$ $\psi_j \in (C^m(K), \|\cdot\|_{m,\infty})$ $\|f\|_{m,\infty} = \sup_{|\alpha| \leq m} \|D^\alpha f\|_\infty$

$\forall m$ fixed $\psi_j \to 0$ in $C^m(K)$ (taking $j \geq m$)

$\|D^\alpha \psi_j\|_\infty \to 0$ $\forall \alpha \Rightarrow \psi_j \to 0$ in $C_c^\infty(U)$: so by ① $T\psi_j \to 0$

$|T\psi_j| = 1 \longrightarrow 0$ CONTRADICTION

DEFINITION OF ORDER

Let T be a distribution. If $\exists p \in \mathbb{N}$ s.t. that $\forall K \subset\subset U$

$\exists C_K > 0 \quad |T\varphi| \leq C_K \sup_{|\alpha| \leq p} \|D^\alpha \varphi\|_\infty$. Then p = ORDER of the DISTRIBUTION

ex 1) μ Radon measure on $\mathbb{R}^n$ $\mu_{|U}(A) = \mu(U \cap A)$ is a Radon measure on U.

We associate to μ a distribution $T_\mu(\varphi) := \int_U \phi \, d\mu$

$\operatorname{supp} \phi \subseteq K \subset\subset U$

$|T_\mu \phi| \leq \int_K |\phi| \, d\mu \leq \|\phi\|_\infty \mu(K)$ $\quad C_K = \mu(K)$

T_μ is a distribution of order 0

$f \in L^1_{loc}(U) \rightsquigarrow T_f(\varphi) = \int_U \phi f \, dx$ is a distribution of order 0.

Obs If T has order k then T can be extended to a linear continuous functional $T: C_c^k(U) \to \mathbb{R}$

where on $C_c^k(U)$ we put the topology: $\phi_n \to \phi$ if $\exists K \subset\subset U$ $\operatorname{supp}\phi_n, \operatorname{supp}\phi \subseteq K$ and $\sup_{|\alpha|\le k} \|D^\alpha\phi_n - D^\alpha\phi\|_\infty \to 0$

So distributions of order 0 are RADON MEASURES on U

T DISTRIBUTION OF ORDER ZERO $\iff$ $\exists\, \mu^+, \mu^-$ RADON MEASURES on $\mathbb{R}^n$ such that $T\phi = \int_U \phi\, d\mu^+ - \int_U \phi\, d\mu^- \quad \forall \phi \in C_c^\infty(U)$

Ex $T: C_c^\infty(\mathbb{R}) \to \mathbb{R}$ $T(\varphi) = \varphi'(0)$ has order 1

$|T\varphi| \le C\|\varphi'\|_\infty \quad \forall \varphi \in C_c^\infty(\mathbb{R}) \Rightarrow \operatorname{order} T \le 1$

Moreover let $\psi \in C_c^\infty(\mathbb{R})$ $\psi'(0) = 1$ $\varphi_n(x) := \psi(nx)$ $\varphi_n'(0) = n$

Let $[-R,R] \supseteq \operatorname{supp}\psi$ then $[-R,R] \supseteq \operatorname{supp}\varphi_n \ \forall n$. $\|\varphi_n\|_\infty = \|\psi\|_\infty$.

$|T\varphi_n| = n \not\le C_R\|\varphi_n\|_\infty \Rightarrow T$ cannot have order 0!

Ex — A distribution of infinite order

$$T : C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\forall \phi \in C_c^\infty(\mathbb{R}) \qquad T(\phi) = \sum_{n=0}^{+\infty} \phi^{(n)}(n) =$$

$$= \phi(0) + \phi'(1) + \phi''(2) + \phi'''(3) + \dots$$

Note that $\forall \phi$ $\sum_{n=0}^{+\infty} \phi^{(n)}(n)$ is finite (since

$\phi^{(n)}(n) = 0 \quad \forall n \notin \operatorname{Supp} \phi$)

Moreover the order of ϕ cannot be finite.