Theorem In $\mathbb{R}^n$ we have that $\mathcal{H}^n_g \equiv \mathcal{H}^n \equiv \mathcal{L}^n$

(the n-Hausdorff measure coincides with Lebesgue)

(NB for $n=1$ already proved)

Corollary (1) H^s is not Radon $\forall\, s < n$.

(proof $K \subset\subset \mathbb{R}^n$ $H^n(K) = |K| > 0 \Rightarrow H^s(K) = +\infty$ $\forall\, s < n$)

2) $H^s \equiv 0$ if $s > n$

(proof $E \in \mathcal{B}(\mathbb{R}^n)$ $|E| < +\infty \Rightarrow H^n(E) = |E| \Rightarrow H^s(E) = 0$ $\forall\, s > n$

if $E \in \mathcal{B}(\mathbb{R}^n)$ $|E| = +\infty$ $E = \cup_i E_i$ $|E_i| < +\infty$ $E_i = E \cap B(0,i)$

$|E_i| < +\infty \Rightarrow H^s(E_i) = 0$ $\forall\, s > n$ $H^s(E) \leq \sum_i H^s(E_i) = 0$

To prove the theorem we pass through an important inequality.

ISODIAMETRIC INEQUALITY $\forall E \in \mathcal{B}(\mathbb{R}^n)$

$$|E| \leq \omega_n \left(\frac{\operatorname{diam}(E)}{2}\right)^n \quad (= \text{volume of the ball with diameter} = \operatorname{diam}(E))$$

that is:
(among sets of given diameter, the ball has maximal volume / among sets of given volume the ball has minimal diameter.

$|E| = \inf \{ \sum_i \operatorname{diam} C_i \quad E \subseteq \cup_i C_i \}$

proof $n = 1$ trivial $|E| \leq \operatorname{diam} E$

$n > 1$ proof by **STEINER SYMMETRIZATION** (method used in several other context in minimization pb)

STEINER SYMMETRIZATION PROCEDURE

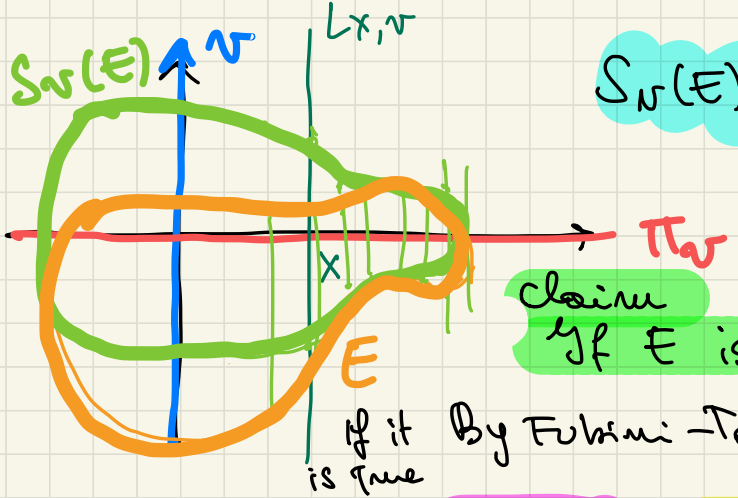
Let $E \subseteq \mathbb{R}^n$. Fix $\nu \in S^{n-1}$ ($|\nu|=1$) "a direction"

π_ν = Hyperplane $\perp \nu$ passing through the origin

$\stackrel{!}{=} \{ x \in \mathbb{R}^n \quad x \cdot \nu = 0 \}$

↑ scalar product

$\forall x \in \pi_\nu \quad L_{x,\nu} = x + t\nu \quad t \in \mathbb{R}$ (straight line $\parallel \nu$ through x)



$$S_\nu(E) = \bigcup_{x \in \pi_\nu} \{ x + t\nu, \ |t| \le \frac{1}{2} \mathcal{H}^1(E \cap L_{x\nu}) \}$$

↓ Steiner symmetrization of E w.r.t. ν

claim: If E is measurable $\to S_\nu(E)$ is measurable

If it is true By Fubini-Tonelli $\boxed{|S_\nu(E)| = |E|}$

claim: $\operatorname{diam} S_\nu(E) \le \operatorname{diam} E$

Claim

if E is measurable $\Rightarrow S_\nu(E)$ is measurable

SUBGRAPH OF A MEASURABLE FUNCTION IS MEASURABLE

Thm $f: \mathbb{R}^n \to [0, +\infty)$ measurable ($\mathcal{L}$-meas.)

$A = \{(x, y) \quad y \le f(x)\}$ is measurable in $\mathbb{R}^{n+1}$

proof: $F: (x,y) \xrightarrow{\text{meas}} (f(x), y) \xrightarrow{\text{Borel}} f(x) - y$ $\quad A = \{F(x,y) \ge 0\}$ measurable

(measurable)

Take $\nu = e_1$ $\quad \mathcal{H}^1 = \mathcal{L}$ on $\mathbb{R}$ $\quad \pi_\nu = \{0\} \times \mathbb{R}^{n-1}$ $\left(\cong \mathbb{R}^{n-1}\right)$

$f: \mathbb{R}^{n-1} \to \mathbb{R}$ $\quad \pi_\nu$ is

By Fubini the map $f: x \longmapsto \mathcal{H}^1(E \cap L_{x\nu})$ is measurable $|E| = \int_{\mathbb{R}^n} \chi_E \, dx = \int_{\mathbb{R}^{n-1}} f(x)\, dx = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(E \cap L_{x\nu})\, dx$

$S_\nu(E) = \left\{(y_1, x) \quad -\frac{f(x)}{2} \le y_1 \le \frac{f(x)}{2}\right\} \Rightarrow$ measurable

Therefore $|E| = |S_\nu(E)|$

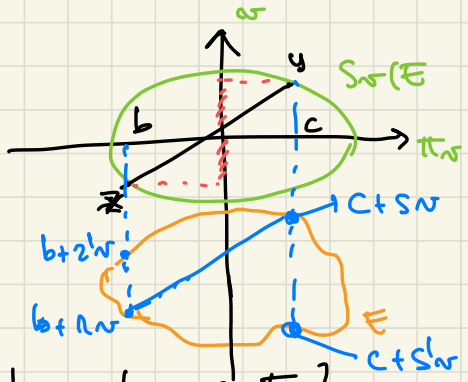
Claim: $\operatorname{diam} S_\nu(E) \le \operatorname{diam} E$.

Take $z, y \in S_\nu(E)$, $|z-y|^2 \ge (\operatorname{diam} S_\nu(E))^2 - \epsilon$ ϵ small

$z \in S_\nu(E)$ $\quad z = X_1 + t\nu$ $\quad b \in \pi_\nu \to t = b\cdot\nu$ $\quad (b\cdot\nu = 0!)$

$z = X_1 + (z\cdot\nu)\nu$

$y \in S_\nu(E)$ $\quad y = X_2 + (y\cdot\nu)\nu$ $\quad c \in \pi_\nu$



$$(\operatorname{diam} S_\nu(E))^2 - \epsilon \le |z-y|^2 = |X_1 - X_2|^2 + |z\cdot\nu - y\cdot\nu|^2$$

$s := \sup\{t \mid X_2 + t\nu \in E\}$ $\quad s' = \inf\{t \mid X_2 + t\nu \in E\}$

$r := \inf\{t \mid X_1 + t\nu \in E\}$ $\quad r' = \sup\{t \mid X_1 + t\nu \in E\}$

$|y\cdot\nu| \le \frac{s-s'}{2}$ $\quad |z\cdot\nu| \le \frac{r'-r}{2}$

Assume without loss of generality $\boxed{s-r \le r'-s'}$

$$|z\cdot\nu - y\cdot\nu| \le \frac{s-s'}{2} + \frac{r'-r}{2} = \frac{s-r}{2} + \frac{r'-s'}{2} \le \frac{r'-s'}{2} + \frac{r'-s'}{2} = r'-s'$$

$$|z-y|^2 = |x_1-x_2|^2 + |t\cdot v - y\cdot v|^2 \le |x_1-y_2|^2 + |r'-s'|^2 =$$

$$= |(x_1+r'v)-(x_2+s'v)|^2 \le (\operatorname{diam} E)^2 \qquad \begin{pmatrix} x_1+r'v\in\overline{E} \\ x_2+s'v\in E \end{pmatrix}$$

$$(\operatorname{diam} S_v(E))^2 - \varepsilon \le (\operatorname{diam} E)^2$$

$\varepsilon\to 0$ $\Rightarrow$ $\operatorname{diam} S_v(E) \le \operatorname{diam}(E)$.

Observ. $S_v(E)$ symmetric w.r.to π_v.

If E is symmetric w.r.to π_w, with $w\perp v$ $(w\cdot v=0)$ then $S_v(E)$ is symmetric w.r.to π_v, π_w.

$E_1 = S_{e_1}(E)$ $\quad e_1=(1,0\dots 0)$ $\quad$ symm w.r.t $\pi_{e_1}=\{x_1=0\}$

$E_2 = S_{e_2}(E_1)$ $\quad\to$ symm w.r.to π_{e_1}, π_{e_2}

$E^* = S_{e_n}(E_{n-1})$ $\to$ symmetric w.r.to $\pi_{e_1}, \pi_{e_2}, \dots \pi_{e_n}$

$|E^*| = |E|$ $\quad \operatorname{diam} E^* \le \operatorname{diam} E$.

E^* is symmetric w.r.t all coordinate axis.
(STEINER SYMMETRIZATION of E).

$$\downarrow$$

$$E^* \subseteq \overline{B\left(0, \frac{\operatorname{diam} E^*}{2}\right)} \subseteq \overline{B\left(0, \frac{\operatorname{diam} E}{2}\right)}$$

$$|E| = |E^*| \leq \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n \qquad \square.$$

Theorem $\mathcal{H}^n(E) = \mathcal{H}^n_\delta(E) = |E| \qquad \forall\, E \in \mathcal{B}(\mathbb{R}^n)$

Step 1 Let E_i closed with $\operatorname{diam} E_i \leq \delta$, $E \subseteq \cup_i E_i$

$$|E| \leq \sum_i |E_i| \underset{\text{(ISODIAM)}}{\leq} \sum_i \omega_n \left(\frac{\operatorname{diam} E_i}{2}\right)^n = \frac{\omega_n}{2^n} \sum_i (\operatorname{diam} E_i)^n$$

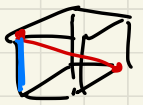
taking the infimum among all possible covers:

$$|E| \leq \mathcal{H}^n_\delta(E) \leq \mathcal{H}^n(E) \qquad \forall \delta > 0.$$

Step 2 Let Q a cube in $\mathbb{R}^n$ $\quad Q = (a,b)^n = (a,b)\times(a,b)\cdots\times(a,b)$

$|Q| = \left(\frac{\operatorname{diam} Q}{\sqrt{n}}\right)^n$ (edge of Q is $\frac{\operatorname{diam} Q}{\sqrt{n}}$)

$\operatorname{diam} Q = \text{edge of } Q \cdot \sqrt{n}$



$H^n_\delta(E) \le \inf\left\{ \frac{\omega_n}{2^n} \sum_i (\operatorname{diam} Q_i)^n \;\middle|\; \begin{array}{l} \operatorname{diam} Q_i \le \delta \\ Q_i \text{ cube} \\ E \subseteq \cup_i Q_i \end{array} \right\} \le$

$\le \inf\left\{ \frac{\omega_n}{2^n} \sum_i n^{\frac{n}{2}} \cdot |Q_i| \right\} \le \inf\left\{ \frac{\omega_n}{2^n} n^{n/2} \sum_i |Q_i| \;\middle|\; \begin{array}{l} Q_i \text{ cube} \\ E \subseteq \cup_i Q_i \\ \operatorname{diam} Q_i \le \delta \end{array} \right\}$

$\le \frac{\omega_n}{2^n} n^{n/2} |E|$

Letting $\delta \to 0^+$ $\quad \forall E \in \mathcal{B}(\mathbb{R}^n)$ $\quad H^n(E) \le \frac{\omega_n}{2^n} n^{n/2} |E| \Rightarrow \boxed{H^n \ll \mathcal{L}}$

Step 3 Fix $\varepsilon > 0$ $\quad Q_i$ cubes of diam $< \delta$ such that

$E \subseteq \cup_i Q_i \qquad |E| \le \sum_i |Q_i| \le |E| + \varepsilon$

By Vitali covering theorem, $\forall\, Q_i^\circ$ (interior of Q_i) there exist a

countable covering of closed disjoint balls of diameter $\leq \delta$

$$|Q_i \setminus \bigcup_k B_i^k| = 0 \qquad \bigcup_i B_i^k \subseteq Q_i^\circ$$

$$\downarrow$$

$$\mathcal{H}^n(Q_i \setminus \bigcup_k B_i^k) = 0 \quad \text{since } \mathcal{H}^n \ll \mathcal{L}, \quad \mathcal{H}^n_\delta(Q_i \setminus \bigcup_k B_i^k) = 0$$

$$\mathcal{H}^n_\delta(E) \leq \sum_i \mathcal{H}^n_\delta(Q_i) = \sum_i \mathcal{H}^n_\delta\Big(\bigcup_k B_i^k\Big) \leq \sum_i \sum_k \mathcal{H}^n_\delta(B_i^k)$$

(σ-subadditivity; σ-subadditivity)

$$\leq \sum_i \sum_k \frac{\omega_n}{2^n} (\operatorname{diam} B_i^k)^n = \sum_i \sum_k |B_i^k| = \sum_i |Q_i| \leq |E| + \varepsilon$$

$\downarrow$ $\operatorname{diam} B_i^k < \delta$!
CHOOSE COVERING OF B_i^k as B_i^k ITSELF

$$\mathcal{H}^n_\delta(E) \leq |E| + \varepsilon \implies \mathcal{H}^n_\delta(E) \leq |E|$$

$$\downarrow \mathcal{H}^n(E) \leq |E|$$

We conclude

$$\boxed{|E| \leq \mathcal{H}^n_\delta(E) \leq \mathcal{H}^n(E) \leq |E|}$$

$\forall E \in \mathcal{B}(\mathbb{R}^n)$