COMPUTABILITY (19/10/2026)

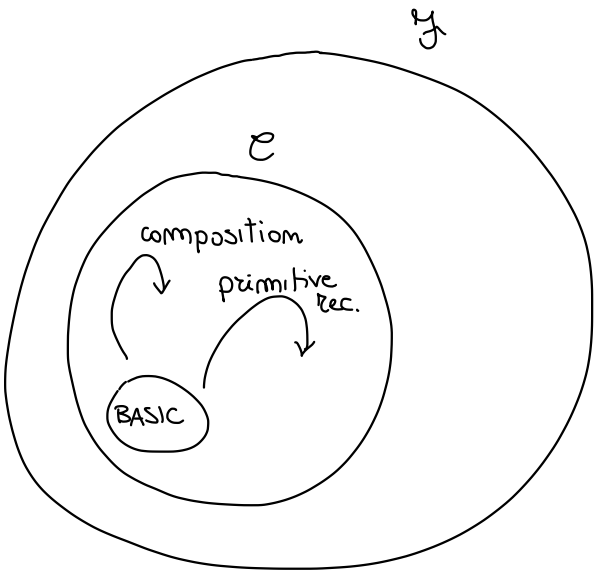
Class $\mathcal{C}$ of URM-computable functions

* contains the BASIC FUNCTIONS

  (a) zero
  (b) successor
  (c) projections

* closed under

  (1) (generalised) composition
  (2) primitive recursion
  (3) (unbounded) minimalisation



Examples

* Sum

$$\begin{cases} x+0 = x \\ x+(y+1) = (x+y)+1 \end{cases}$$

$f(x) = x = U_1^1(x)$

$g(x,y,z) = z+1$

* Product

$$\begin{cases} x*0 = 0 \\ x*(y+1) = (x*y)+x \end{cases}$$

$f(x) = 0$

$g(x,y,z) = z+x$

* Exponential x^y

$$\begin{cases} x^0 = 1 \\ x^{y+1} = (x^y)*x \end{cases}$$

* Predecessor $y \dot{-} 1$

$$\begin{cases} 0 \dot{-} 1 = 0 \\ (y+1) \dot{-} 1 = y \end{cases}$$

* **Difference** $x \dot{-} y = \begin{cases} 0 & \text{if } x < y \\ x - y & \text{if } x \geq y \end{cases}$

$x \dot{-} 0 = x$

$x \dot{-} (y+1) = (x \dot{-} y) \dot{-} 1$

($\dot{-}$: difference; $\dot{-} 1$: predecessor)

* **Sign** $sg(y) = \begin{cases} 0 & \text{if } y = 0 \\ 1 & \text{if } y > 0 \end{cases}$

$\begin{cases} sg(0) = 0 \\ sg(y+1) = 1 \end{cases}$

* **Sign negated** $\overline{sg}(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$

EXERCISE $= 1 \dot{-} sg(y)$

* **minimum**

$\min(x,y) = x \dot{-} (x \dot{-} y)$

$x \dot{-} y = 0$ ← $x \leq y$

$x \dot{-} y > 0$ ← $x > y$

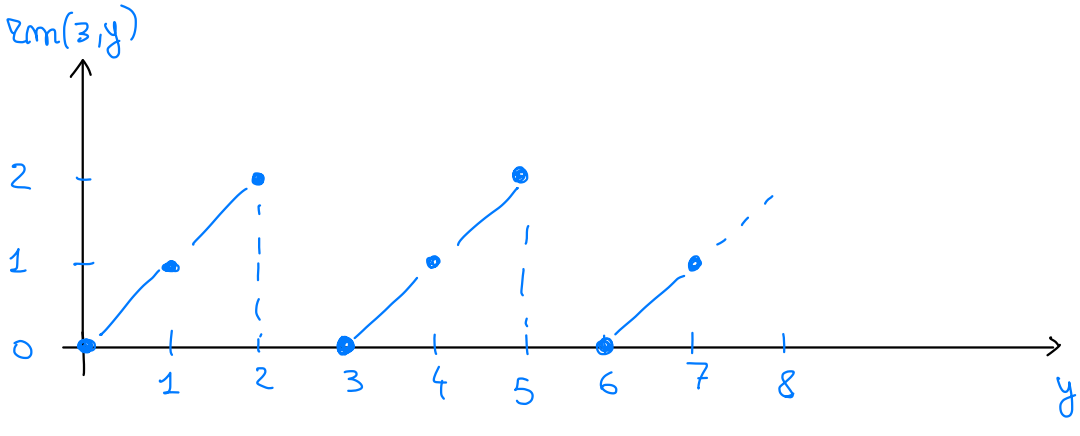
$x - (x - y) = y$

* **maximum**

$\max(x,y)$

* $rm(x,y)$ = remainder of y divided by x

(conventionally $rm(0,y) = y$)



$$\begin{cases} rm(x,0) = 0 \\ rm(x,y+1) = \begin{cases} rm(x,y)+1 & \text{if } rm(x,y)+1 < x \\ 0 \end{cases} \end{cases}$$

$$= (rm(x,y)+1) * \underbrace{sg(x \dot{-} (rm(x,y)+1))}$$

0 if $rm(x,y)+1 \geq x$

1 if $rm(x,y)+1 < x$

* $qt(x,y) \equiv y \text{ div } x$ (convention $qt(0,y) = 0$)

(exercise) SOLUTION:

$qt(x,0) = 0$

$$qt(x,y+1) = \begin{cases} qt(x,y)+1 & \text{if } rm(x,y+1) = 0 \\ qt(x,y) & \text{otherwise} \end{cases}$$

$$= qt(x,y) + \overline{sg}(rm(x,y+1))$$

* OBSERVATION: Define a computable function by cases

Let $f_1, \dots, f_m : \mathbb{N}^k \to \mathbb{N}$ function computable total

$Q_1(\vec{x}), \dots, Q_m(\vec{x}) \subseteq \mathbb{N}^k$ decidable predicates $\forall \vec{x}\ \exists!\, j\ Q_j(\vec{x})$

Then $f: \mathbb{N}^k \to \mathbb{N}$

$$f(\vec{x}) = \begin{cases} f_1(\vec{x}) & \text{if } Q_1(\vec{x}) \\ f_2(\vec{x}) & \text{if } Q_2(\vec{x}) \\ \vdots & \\ f_m(\vec{x}) & \text{if } Q_m(\vec{x}) \end{cases}$$

Then f is computable

proof

$$f(\vec{x}) = f_1(\vec{x}) * \chi_{Q_1}(\vec{x}) + f_2(\vec{x}) * \chi_{Q_2}(\vec{x}) + \dots + f_m(\vec{x}) * \chi_{Q_m}(\vec{x})$$

computable by hyp.

$\chi_{Q_1}(\vec{x})$: 1 if $Q_1(\vec{x})$, 0 otherwise

$f_1(\vec{x}) * \chi_{Q_1}(\vec{x})$: $f_1(\vec{x})$ if $Q_1(\vec{x})$, 0 otherwise

computable

computable by composition

Note : $m = 2$, $k = 1$

$f_1(x) = x$ $\quad \forall x$ $\quad Q_1(x) =$ "x is even"

$f_2(x)$ ↑ $\quad Q_2(x) =$ "x is odd"

$$f(x) = f_1(x) \cdot \chi_{Q_1}(x) + f_2(x) \cdot \chi_{Q_2}(x) \quad \uparrow \forall x$$

$\forall x$

$$\not\exists \begin{cases} x & \text{if } x \text{ is even } (Q_1) \\ \uparrow & \text{if } x \text{ is odd} \end{cases}$$

* Algebra of decidability

Let $Q_1(\vec{x}), Q_2(\vec{x}) \subseteq \mathbb{N}^k$ be decidable predicates. Then

1) $\neg Q_1(\vec{x})$

2) $Q_1(\vec{x}) \wedge Q_2(\vec{x})$

3) $Q_1(\vec{x}) \vee Q_2(\vec{x})$

1) $\chi_{Q_1}(\vec{x}) = 0$

$$\chi_{\neg Q_1}(\vec{x}) = \begin{cases} 1 & \text{if } Q_1(\vec{x}) \text{ false} \\ 0 & \text{if } Q_1(\vec{x}) \text{ true} \end{cases} = \overline{sg}(\chi_{Q_1}(\vec{x}))$$

$\chi_{Q_1}(\vec{x}) = 1$

2) $\chi_{Q_1 \wedge Q_2}(\vec{x}) = \chi_{Q_1}(\vec{x}) * \chi_{Q_2}(\vec{x})$

3) $\chi_{Q_1 \vee Q_2}(\vec{x}) = sg(\chi_{Q_1}(\vec{x}) + \chi_{Q_2}(\vec{x}))$

□

* Bounded Sum / Product

Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ $\quad f(\vec{x}, z)$ $\quad$ total computable

define $h : \mathbb{N}^{k+1} \to \mathbb{N}$

$$h(\vec{x}, y) = f(\vec{x}, 0) + f(\vec{x}, 1) + \dots + f(\vec{x}, y-1)$$

$$= \sum_{z<y} f(\vec{x}, z)$$

total computable

$$\begin{cases} h(\vec{x}, 0) = 0 \\ h(\vec{x}, y+1) = h(\vec{x}, y) + f(\vec{x}, y) \end{cases}$$

computable by primitive recursion

* $\prod_{z<y} f(\vec{x}, z)$

$$\prod_{z<0} f(\vec{x}, z) = 1$$

$$\prod_{z<y+1} f(\vec{x}, z) = \left(\prod_{z<y} f(\vec{x}, z) \right) * \left(f(\vec{x}, y) \right)$$

* Bounded Quantification

Given $Q(\vec{x}, z) \subseteq \mathbb{N}^{k+1}$ decidable

$$P_1(\vec{x}, y) \equiv \forall z < y \quad Q(\vec{x}, z)$$
$$P_2(\vec{x}, y) \equiv \exists z < y \quad Q(\vec{x}, z)$$

EXERCISE : decidable

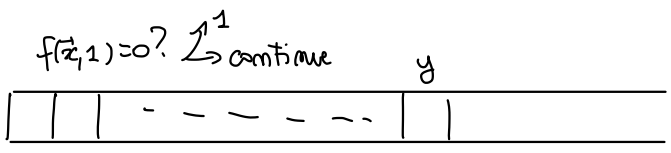
* Bounded minimalisation

Given $f: \mathbb{N}^{k+1} \to \mathbb{N}$ total

define $h: \mathbb{N}^{k+1} \to \mathbb{N}$

$$h(\vec{x}, y) = \begin{cases} \text{minimum } z \text{ s.t. } z<y \text{ and } f(\vec{x}, z) = 0 \\ (\text{if there is such a } z) \\ y \quad \text{otherwise} \end{cases}$$

$$= \mu z < y . \quad f(\vec{x}, z)$$ ✗

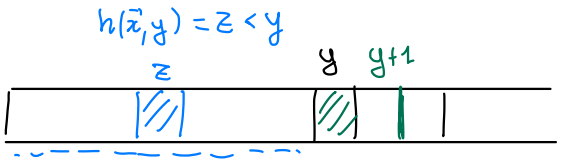


$f(\vec{x}, 0) = 0? \to 0$
↘ cont

OBSERVATION: If f is computable then $\mu z<y.\ f(\vec{x},z)$ computable

Let $h(\vec{x},y) = \mu z<y.\ f(\vec{x},z)$

$h(\vec{x},0) = 0$



$$h(\vec{x},y+1) = \begin{cases} \text{if } h(\vec{x},y) < y & \rightsquigarrow \text{ result } h(\vec{x},y) \\ \text{if } h(\vec{x},y) = y & \rightsquigarrow \begin{cases} f(\vec{x},y) = 0 & \rightsquigarrow \text{ result } y = h(\vec{x},y) \\ f(\vec{x},y) \neq 0 & \rightsquigarrow \text{ result } y+1 \end{cases} \end{cases}$$

$y+1 = h(\vec{x},y)+1$

$$= h(\vec{x},y) + \overline{sg}(y \dot{-} h(\vec{x},y)) * sg(f(\vec{x},y))$$

1 if $h(\vec{x},y) = y$ and $f(\vec{x},y) \neq 0$

0 otherwise

OBSERVATION: The following functions are computable

* $div : \mathbb{N}^2 \to \mathbb{N}$

$$div(x,y) = \begin{cases} 1 & \text{if } x \text{ divides } y \\ 0 & \text{otherwise} \end{cases}$$

$$= \overline{sg}(rm(x,y))$$

* $D(x)$ = number of divisors of x

$$= \sum_{y \leq x} div(y,x)$$

↑ $y < x+1$

* $Pr(x) = \begin{cases} 1 & x \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$

x prime iff it has exactly two divisors (1 and x, different)

$$Pr(x) = \overline{sg}(|D(x) - 2|)$$

↑ $|x-y| = (x \dot{-} y) + (y \dot{-} x)$

* p_x = x-th prime number

$p_0 = 0 \quad p_1 = 2 \quad p_2 = 3 \quad p_3 = 5 \quad p_4 = 7 \quad p_5 = 11 \quad p_6 = 13$

$$\begin{cases} p_0 = 0 \\ p_{x+1} = \mu z \end{cases} \qquad z \text{ prime and } z > p_x$$

$$= \mu z \leq \left(\prod_{i=1}^{x} p_i\right) + 1 \,.\, \overline{sg}\left(p_2(z) \cdot (z = p_x)\right)$$

since $p_{x+1} \leq \left(\prod_{i=1}^{x} p_i\right) + 1$

let p prime factor of ↑ — p divides $\left(\prod_{i=1}^{x} p_i\right) + 1$

$\Rightarrow$ $p \neq p_i \quad \forall i = 1 \dots x$

otherwise p divides $\prod_{i=1}^{x} p_i$

$\Rightarrow$ p divides 1 $\Rightarrow$ $p = 1$ not a prime number

so $p_{x+1} \leq p \leq \left(\prod_{i=1}^{x} p_i\right) + 1$

as desired □

* $(x)_y$ ≡ exponent of p_y in the prime factorisation of x

$140 = 2^2 \cdot 5 \cdot 7$

$= 2^2 \cdot 3^0 \cdot 5^1 \cdot 7^1 \cdot 11^0 \cdot \dots$

$(140)_1$ = exponent of p_1 ($= 2$) in prime decomp. of 140 = 2

$(140)_2$ = exponent of $p_2 = 3$ = 0

$(140)_3$ = 1

$(140)_4$ = 1

$$(x)_y = \max z \quad p_y^z \text{ divides } x$$

$$= \min z \quad p_y^{z+1} \text{ does not divide } x$$

$$= \mu z \leq x \;.\; \mathrm{div}(p_y^{z+1}, x)$$

EXERCISE : All the functions obtained from basic functions using composition and primitive recursion are total