HAUSDORFF MEASURES

(Ch 2 Evans-Gariepy, Appendix Folland)

Introduce Borel measures on $\mathbb{R}^n$ which "can see" set of "low dimension".

$s \geq 0$ $\quad \omega_s = \dfrac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$ $\quad \omega_n = |B(0,1)|$

$\omega_0 = 1 \quad \omega_1 = \dfrac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} = 2$

- $\operatorname{diam} E = \operatorname{diam} \bar{E}$
- $\operatorname{diam} E = \operatorname{diam}(\operatorname{Co} E)$
- $E \subseteq A_\varepsilon$ open $\quad \operatorname{diam} E + \varepsilon$

$\delta > 0$ $\quad E \subseteq \mathbb{R}^n$ subset $\quad \operatorname{diam} E := \sup_{x,y \in E} |x-y|$

$$H^s_\delta(E) := \frac{\omega_s}{2^s} \inf\left\{ \sum_1^\infty i \, [\operatorname{diam} E_i]^s \, , \; E \subseteq \bigcup_{i=1}^\infty E_i \; \operatorname{diam} E_i \leq \delta \right\}$$

DEFINITION ↙ DOES NOT CHANGE if we restrict to either E_i open or E_i closed or E_i convex

$H^s_\delta : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$

IT CHANGES if we restrict to balls E_i (Spherical Hausdorff measure)



equilateral triangle $\quad \operatorname{diam} T = \text{side} < \operatorname{diam} B$

Fix $s \geq 0$ $\quad \delta_1 > \delta_2$ $\quad (\operatorname{diam} E_i \leq \delta_2 < \delta_1)$

$$\downarrow$$

$$H^s_{\delta_1}(E) \leq H^s_{\delta_2}(E)$$

$$H^s(E) = \lim_{\delta \to 0^+} H^s_\delta(E) = \sup_{\delta > 0} H^s_\delta(E)$$

(s-dimensional Hausdorff measure)

- Observe that $A, B \subseteq \mathbb{R}^n$ $\quad H^s_\delta(A \cup B) = H^s_\delta(A) + H^s_\delta(B)$ if $\operatorname{dist}(A,B) > 4\delta$

Proposition (NO PROOF)

1) H^s_δ is a OUTER MEASURE on $\mathbb{R}^n$

2) H^s is a OUTER MEASURE on $\mathbb{R}^n$

$$\downarrow$$

$H^s(A \cup B) = H^s(A) + H^s(B)$ if $\operatorname{dist}(A,B) > 0$

Carathéodory: H^s is a Borel measure

ALSO Borel REGULAR : $\forall A \subseteq \mathbb{R}^n \quad \exists E \in \mathcal{B}(\mathbb{R}^n) \; A \subseteq E \quad H^s(A) = H^s(E)$

NB H^s is NOT RADON MEASURE (except that for $s = n$ $\leftarrow \dim \mathbb{R}^n$)

(we will prove it)

Proposition

1) $\mathcal{H}^0$ is the COUNTING MEASURE $\quad \mathcal{H}^0(E) = \#\{x \in E\} = \text{card } E$

$\mathcal{H}^0_\delta(E) = \inf_\delta \{\#E_i \,,\ \cup_i E_i \supseteq E \ \text{diam } E_i \le \delta\}$

2) $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}^1$ i.e. $\mathcal{H}^1_\delta(A) = \mathcal{H}^1(A) = |A| \ \forall A \in \mathcal{B}(\mathbb{R})$

Let $A \subseteq \mathbb{R}$ $\quad |A| = \inf \{\sum_i (\text{diam } C_i), A \subseteq \cup_i C_i\}$

$\le \inf \{\sum_i (\text{diam } C_i)^1, \ A \subseteq \cup_i C_i \ \text{diam } C_i \le \delta\}$

$= H^1_\delta(A) \Rightarrow |A| = \mathcal{L}^1(A) \le H^1_\delta(A) \ \forall \delta > 0 \Rightarrow |A| \le \mathcal{H}^1(A)$

Fix $\delta > 0$
$I_k = [k\delta, (k+1)\delta]$ $\quad \text{diam}(C_i \cap I_k) \le \delta \quad \forall k \in \mathbb{Z}$

$\text{diam } C_i \ge \sum_k (\text{diam } C_i \cap I_k)$ [exercise]

$|A| = \inf \{\sum_i (\text{diam } C_i), A \subseteq \cup_i C_i\} \ge \inf \{\sum_i \sum_k (\text{diam}(C_i \cap I_k)) \ : \ A \subseteq \cup_i C_i\}$

$\geq \inf \left(\sum_i (\operatorname{diam} B_i)^1 \right)$ $\operatorname{diam} B_i \leq \delta$ $\bigcup_i B_i \supseteq A \Big\} = H^1_\delta(A)$

so $\forall \delta > 0$ $\quad \mathcal{H}^1_\delta(A) \leq |A| \leq \mathcal{H}^1_\delta(A)$ $\quad \boxed{H^1_\delta \equiv \mathcal{H}^1 \equiv \mathcal{L}^1 \text{ on } \mathbb{R}}$

(on $\sum_k \operatorname{diam}(C_i \cap I_k) \leq \operatorname{diam} C_i$. If $\operatorname{diam} C_i = +\infty$ ok, if $\operatorname{diam} C_i = R \rightarrow C_i \subseteq [0, R]$ up to translation

$C_i = \bigcup_{k=0}^{[R]+1} (C_i \cap I_k)$ $\quad R \geq \sum_{k=\dots} \operatorname{diam}(C_i \cap I_k)$.

(3) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}(A)$ $\quad \forall \lambda > 0$ $\quad \lambda A = \{\lambda x \ x \in A\}$

$\mathcal{H}^s(FA) \leq K^s \mathcal{H}^s(A)$ $\quad F : \mathbb{R}^m \rightarrow \mathbb{R}^m$

$\frac{F(x) - F(y)}{|x-y|} \leq K \quad \forall x \neq y$

Proposition Let $0 \le s_1 < s_2$. $\exists\, \bar{d} \ge 1$ (dimensional constant)

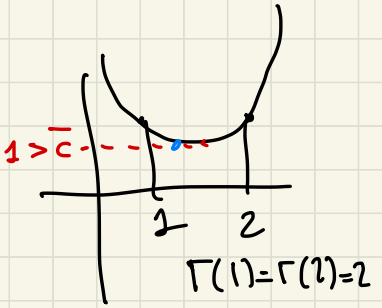
$$H^{s_2}_\delta(E) \le \bar{d}\, \delta^{s_2 - s_1} H^{s_1}_\delta(E) \qquad \forall E \subseteq \mathbb{R}^n \qquad \delta > 0$$

Proof Let $E \subseteq \mathbb{R}^n$ E_i with $\operatorname{diam} E_i < \delta$ $E \subseteq \bigcup_i E_i$

$$\frac{\omega_{s_2}}{2^{s_2}} \sum_i (\operatorname{diam} E_i)^{s_2} = \left(\frac{\sqrt{\pi}}{2}\right)^{s_2} \cdot \frac{1}{\Gamma\left(\frac{s_2}{2}+1\right)} \sum_i (\operatorname{diam} E_i)^{s_1} (\operatorname{diam} E_i)^{s_2 - s_1}$$

$$\le \left(\frac{\sqrt{\pi}}{2}\right)^{s_1} \frac{1}{\bar{c}} \frac{1}{\Gamma\left(\frac{s_1}{2}+1\right)} \delta^{s_2 - s_1} \sum_i (\operatorname{diam} E_i)^{s_1}$$

$\frac{1}{\bar{c}} = \bar{d} \ge 1$



$\frac{\sqrt{\pi}}{2} < 1 \Rightarrow \left(\frac{\sqrt{\pi}}{2}\right)^{s_2} \le \left(\frac{\sqrt{\pi}}{2}\right)^{s_1}$

$\Gamma(t)$ is increasing for $t \ge 2$. $\Gamma(1) = \Gamma(2) = 1$ since $\Gamma = \bar{c} = \Gamma(t) < 1$ $t \ge 0$, $t \in (1,2)$

so if $s_2 \ge 2$, $\frac{s_2}{2}+1 \ge 2 \Rightarrow \Gamma\left(\frac{s_2}{2}+1\right) \ge \Gamma\left(\frac{s_1}{2}+1\right)$

$0 < s_1 \le s_2 < 2 \Rightarrow \Gamma\left(\frac{s_2}{2}+1\right) \ge \bar{c} \ge \bar{c}\,\Gamma\left(\frac{s_1}{2}+1\right)$ since $\Gamma\left(\frac{s_1}{2}+1\right) \le 1$, $1 < \frac{s_1}{2}+1 < 2$

$\delta > 0$ $\qquad H_\delta^{s_2}(E) \le C\,\delta^{s_2 - s_1} H_\delta^{s_1}(E) \qquad \delta > 0$

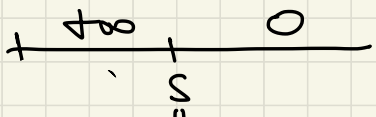
$\downarrow \delta \to 0 \qquad \downarrow \delta \to 0 \qquad \downarrow \delta \to 0$

$H^{s_2}(E) \le 0 \cdot H^{s_1}(E)$

so if $\quad H^{s_2}(E) > 0 \Rightarrow H^{s_1}(E) = +\infty$

$H^{s_1}(E) < +\infty \Rightarrow H^{s_2}(E) = 0$

$(s_2 > s_1)$



$\dim_{\mathcal{H}}(E)$ = Hausdorff dimension of E =

$= \inf\{s \ge 0 \,,\ H^s(E) = 0\} = \sup\{s \ge 0 \ \ H^s(E) = +\infty\}$

NB It may happen that $\dim_{\mathcal{H}}(E) = r > 0$ [$H^r(E) = 0$ or $H^r(E) = +\infty$

(So it is not necessary that $H^r(E) \in (0, +\infty)$

Hausdorff dimension can also be non integer.

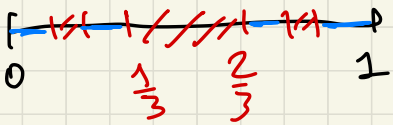
Ex Cantor set in $\mathbb{R}$ is a set with Hausdorff dim. between $0,1$.

$C_0 = [0,1]$ $\quad C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$ $\quad C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

$$C_{n+1} = \frac{1}{3} C_n \cup \left(\frac{1}{3} C_n + \frac{2}{3}\right)$$

C_n closed $\forall n$, and contains 2^n intervals of length $\frac{1}{3^n}$ (with distance between intervals $\geq \frac{1}{3^n}$)

$C = \bigcap_{n=0}^{+\infty} C_n$ is the CANTOR SET (CLOSED)



at every step $n \geq 1$ I remove 2^{n-1} intervals of length $\frac{1}{3^n}$

$$|C| = 1 - \sum_{n=1}^{+\infty} \frac{2^{n-1}}{3^n} = 1 - \sum_n \left(\frac{2}{3}\right)^n \frac{1}{3} = 1 - \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 0$$

$|C| = 0 \Rightarrow \dim_{\mathcal{H}}(C) \leq 1$ (since $\mathcal{H}^1(C) = 0$)

Since $C = \frac{1}{3}C \cup \left(\frac{1}{3}C + \frac{2}{3}\right)$ DISJOINT

$\mathcal{H}^s(C) = 2\mathcal{H}^s\left(\frac{1}{3}C\right) = 2\frac{1}{3^s}\mathcal{H}^s(C)$

if $\exists s$ s.t. $\mathcal{H}^s(C) \neq 0, +\infty \Rightarrow 2\frac{1}{3^s} = 1 \Rightarrow \boxed{s = \log_3 2 = \frac{\log 2}{\log 3}}$

In particular for $s \neq \log_3 2 \Rightarrow \mathcal{H}^s(C) = 0$ or $\mathcal{H}^s(C) = +\infty$.

① take $\delta = \frac{1}{3^n}$ $s > 0$ C_n is a cover of C $(C_n^i)_{i=1,\dots,2^n}$

$$\mathcal{H}^s_{\left(\frac{1}{3}\right)^n}(C) \leq \frac{\omega_s}{2^s} \sum_1^{2^n} \underbrace{\left(\operatorname{diam} C_n^i\right)^s}_{\left(\frac{1}{3^n}\right)^s} = \frac{\omega_s}{2^s}\left(\frac{1}{3^n}\right)^s \cdot 2^n = \frac{\omega_s}{2^s}\left(\frac{2}{3^s}\right)^n$$

$\left(\frac{1}{3}\right)^n \to 0$ $(n \to +\infty) \Rightarrow \mathcal{H}^s(C) \leq \frac{\omega_s}{2^s} \lim_n \left(\frac{2}{3^s}\right)^n = 0$ if $\frac{2}{3^s} < 1$

$\boxed{s > \log_3 2}$

$\Rightarrow \mathcal{H}^s(C) = 0 \ \forall\, s > \log_3(2) \Rightarrow \dim_{\mathcal{H}}(C) \leq \log_3(2)$

Moreover if $s = \log_3 2$

$$\mathcal{H}^{\log_3 2}(C) \leq \frac{\omega_{\log_3 2}}{2^{\log_3 2}}$$

Reverse implication ($\dim_{\mathcal{H}}(C) \geq \log_3 2$) is MUCH MORE DIFFICULT (ref Falconer, Geometry of fractal sets)

one proves $\mathcal{H}^{\log_3 2}(C) \geq \frac{\omega_{\log_3 2}}{2^{\log_3 2}}$

$\Rightarrow \dim_{\mathcal{H}}(C) = \log_3 2$

and $\mathcal{H}^{\log_3 2}(C) = \frac{\omega_{\log_3 2}}{2^{\log_3 2}}$