Ex $L^1(\mathbb{R})$

$$f_n = \begin{cases} n - n^2|x| & x \in (-\frac{1}{n}, \frac{1}{n}) \\ 0 \end{cases}$$



$\|f_n\|_{L^1} = 1 \quad \forall n$

Let $\phi \in L^\infty(\mathbb{R})$ take $\phi \in C_c(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} f_n \phi \, dx = \int_{-1/n}^{1/n} (n - n^2|x|)\,\phi(x)\,dx = \int_{-1}^{1} n\,(1-|y|)\,\phi\left(\frac{y}{n}\right) dy \cdot \frac{1}{n} =$$

$y = nx$

$$= \int_{-1}^{1} (1-|y|)\,\phi\left(\frac{y}{n}\right) dy \longrightarrow \phi(0) = \int \phi \, d\delta_0 \neq \int \phi f \, dx \quad \forall f$$

f_n "converge" in weak sense (w.r.t. L^∞) to δ_0?

$\downarrow \delta_0 \perp \mathcal{L}$ (NOT a density)

NOT A REAL WEAK CONVERGENCE in L^1 SINCE THE LIMIT IS NOT in L^1 !!

Idea was to embed L^1 in a larger space where this kind of weak* compactness holds.

RIESZ THEOREM (Riesz 1909, Markov 1938, Kakutani 1941)

SCALAR VERSION 3 versions

① Let $T: C_c(\mathbb{R}^n) \to \mathbb{R}$ linear functional and POSITIVE ($Tf \geq 0 \ \forall f \geq 0$)

$\exists \mu$ Radon measure such that

$$Tf = \int_{\mathbb{R}^n} f \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n)$$

(observe viceversa (trivially true!))

proof (sketch) ① T positive $\Rightarrow$ $\forall K \subset\subset \mathbb{R}^n$ compact $\exists c_K > 0$ such that $|Tf| \leq c_K \|f\|_\infty$ for all $f \in C_c(\mathbb{R}^n)$ $\operatorname{supp} f \subset K$.

proof Let $\phi \in C_c^\infty(\mathbb{R}^n)$ $\phi \equiv 1$ on K $0 \leq \phi(x) \leq 1$

if $f \in C_c(\mathbb{R}^n)$ $\operatorname{supp} f \subset K \Rightarrow -\|f\|_\infty \phi \leq f \leq \|f\|_\infty \phi \Rightarrow$ by linearity and positivity $-T(\phi)\|f\|_\infty \leq T(f) \leq \|f\|_\infty T(\phi)$ $c_K = T(\phi) \geq 0$.

② U open set $\mu(U) := \sup \{ T(f) \ : \ f \in C_c(\mathbb{R}^n) \ 0 \leq f \leq 1 \ \operatorname{supp} f \subseteq U \}$

$\mu(C) = \inf [\mu(U) \quad U \text{ open} \quad C \subseteq U]$ μ outer measure.

(then prove $\mu(K) = \inf \{ T(f), \; f \geq 1 \text{ on } K, \; 0 \leq f \leq 1, \; f \in C_c(\mathbb{R}^n) \}$, $\Sigma_\mu \supset \mathcal{B}$ (Carathéodory)

$\mu(\mathbb{R}^n) = \sup\{T(f) \quad f \in C_c \; 0 \leq f \leq 1\}$

(2) Let $T : C_c(\mathbb{R}^n) \to \mathbb{R}$ linear s.t. that $\forall K \subset \mathbb{R}^n \; \exists C_K > 0$

$|Tf| \leq C_K \|f\|_\infty \quad \forall f \in C_c(\mathbb{R}^n) \quad \operatorname{supp} f \subseteq K.$

$\exists \mu = \mu^+ - \mu^-$ signed Radon measure

$$Tf = \int_{\mathbb{R}^n} f \, d\mu^+ - \int_{\mathbb{R}^n} f \, d\mu^-$$

$T^+ f = T^+ f^+ - T^+ f^-$
$T^- f = T^- f^- - T^- f^+$

Proof $Tf = T^+ f - T^- f$ T^+, T^- positive and linear
$f \geq 0$ $T^+ f = \sup\{Tg, \; 0 \leq g \leq f, \; g \in C_c(\mathbb{R}) \} \geq Tf, \geq 0$ $T^- f = T^+ f - Tf$

(3) If $T : C_c(\mathbb{R}^n) \to \mathbb{R}$ linear, $\exists C > 0 \; |Tf| \leq C \|f\|_\infty$

$\Rightarrow$ T can be extended to $T : (C_0(\mathbb{R}^n), \|\cdot\|_\infty) \to \mathbb{R}$ linear continuous.

Observe that $T \in (C_0(\mathbb{R}^n), \|\cdot\|_\infty)^*$

$\mu^+(\mathbb{R}^n) \le C$ $\quad \mu^-(\mathbb{R}^n) \le C$ and $\boxed{\|T\| = \mu^+(\mathbb{R}^n) + \mu^-(\mathbb{R}^n)}$

So $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)^*$ = signed Radon measures which are finite. ($\mu^+(\mathbb{R}^n) < +\infty$ $\mu^-(\mathbb{R}^n) < +\infty$)

$(C_0(U), \|\cdot\|_\infty)^*$ = signed Radon measures restricted to U with $\mu^+(U) < +\infty$ $\mu^-(U) < +\infty$

$\downarrow$

$\mu_{\lfloor U}(A) = \mu(A \cap U)$

$C_c(\mathbb{R}^n), \|\cdot\|_\infty$ not a Banach space (not closed)

μ signed Radon measures are linear continuous functional on $C_c(\mathbb{R}^n)$

[if we put in $C_c(\mathbb{R}^n)$ a weaker topology

$f_k \to f$ in $C_c(\mathbb{R}^n)$ if

$\exists K \subset\subset \mathbb{R}^n$ $\operatorname{supp} f_k, \operatorname{supp} f \subseteq K$

$\|f_k - f\|_\infty \to 0$.

Observe that $L^1(\mathbb{R}^n) \subseteq \{$signed Radon measures which are finite$\}$

$f \in L^1(\mathbb{R}^n)$ $\quad f = f^+ - f^-$ $\quad \mu^+ \ll \mathcal{L}$ with density f^+

$\mu^- \ll \mathcal{L}$ " " f^-

$$f \rightsquigarrow \mu_f(A) = \int_A f^+ dx - \int_A f^- dx = \mu_f^+(A) - \mu_f^-(A)$$

$$\|f\|_{L^1} = \mu^+(\mathbb{R}^n) + \mu^-(\mathbb{R}^n)$$

$L^1(\mathbb{R}^n) = \{$signed Radon measures which are finite and absolutely continuous w.r.t. $\mathcal{L}^n\}$

$L^1_{loc}(\mathbb{R}^n) = \{$signed Radon measure which are abs. cont. w.r.t. $\mathcal{L}^n\}$

$\Rightarrow$ L^1 is contained in the dual of a separable space which is $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)$ (SEP. DUE TO STONE-WEIERSTRASS THM)

Weak* compactness in the space of Radon measure.

Let μ_k be a sequence of (signed) finite Radon measures on $\mathbb{R}^n$ with $\mu_k(\mathbb{R}^n) \le C$ (equibounded)
then $\exists\ \mu_{k_m}$ and μ signed finite Radon measure such that $\forall \phi \in C_0(\mathbb{R}^n)$ $\int_{\mathbb{R}^n} \phi \, d\mu_{n_k} \to \int_{\mathbb{R}^n} \phi \, d\mu$

Local version Let μ_n be a sequence of signed Radon measures such that $\forall K \subset\subset \mathbb{R}^n$ $\mu_n(K) \le C_K$.
$\exists\ \mu_{n_k}, \mu$ signed Radon measure s.t.
$\forall \phi \in C_c(\mathbb{R}^n)$ $\int_{\mathbb{R}^n} \phi \, d\mu_{n_k} \to \int_{\mathbb{R}^n} \phi \, d\mu$

Version for L^1 Let $f_n \in L^1(\mathbb{R}^n)$ $\|f_n\|_{L^1} \le C \Rightarrow \exists f_{n_k}$

$\exists \mu$ signed finite Radon measure s.t. that

$\forall \phi \in C_0(\mathbb{R}^n) \subseteq L^\infty \quad \int_{\mathbb{R}^n} \phi \cdot f_{n_k} dx \longrightarrow \int_{\mathbb{R}^n} \phi \, d\mu$

(ex $f_n \to \delta_{x_0}$)

Also local version

$f_n \in L^1_{loc}$ $\|f_n\|_{L^1(K)} \le C_K \Rightarrow \exists f_{n_k} \exists \mu$

$\forall \phi \in C_c(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} \phi f_{n_k} dx \to \int_{\mathbb{R}^n} \phi \, d\mu$

VECTORIAL VERSION of RIESZ

$T : C_c(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ LINEAR s.t. that $\forall K \subset \mathbb{R}^n \ \exists C_K$

$|Tf| \le C_K \|f\|_\infty \quad \forall f \in C_c \quad \operatorname{supp} f \subseteq K$

$\exists \mu$ Radon measure (VARIATION MEASURE)

$\exists \sigma : \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable $|\sigma(x)| = 1$ for μ-a.e. x.

such that $Tf = \int_{\mathbb{R}^n} \underbrace{f \cdot \sigma}_{\text{SCALAR PRODUCT}} \, d\mu \quad \forall f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$