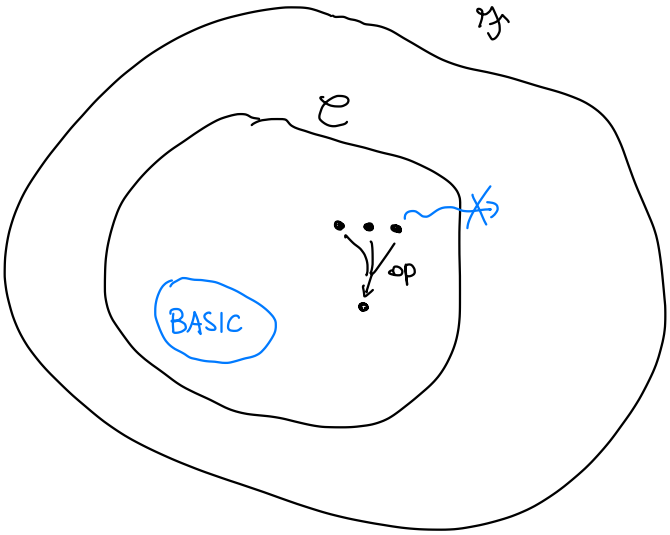
COMPUTABILITY (13/10/2026)

* Generation of computable functions



$\mathcal{C}$ is closed under

- composition
- primitive recursive
- unbounded minimalisation

* BASIC FUNCTIONS

$\vec{x} = (x_1, \dots, x_k)$

①	constant zero	$z : \mathbb{N}^k \to \mathbb{N}$	$z(\vec{x}) = 0 \quad \forall \vec{x} \in \mathbb{N}^k$
②	successor	$s : \mathbb{N} \to \mathbb{N}$	$s(x) = x+1 \quad \forall x \in \mathbb{N}$
③	projection	$U^k_j : \mathbb{N}^k \to \mathbb{N}$	$U^k_j(\vec{x}) = x_j \quad \forall \vec{x} \in \mathbb{N}^k$

They are in $\mathcal{C}$ as they are computed by

① $z(1)$

② $s(1)$

③ $T(j,1)$

* Notation

given a program P

- $\rho(P) = \max\{n \mid \text{register } R_n \text{ is referred in } P\}$
- $\ell(P)$ = number of instructions in P
- P is in standard form if whenever it terminates, it does at instruction index $\ell(P)+1$
- concatenation : given P, Q programs

$$\begin{matrix} P \\ Q \end{matrix} \rightsquigarrow \begin{matrix} P \\ Q' \end{matrix} \leftsquigarrow J(m,m,t) \rightsquigarrow J(m,m,t+\ell(P))$$

- given a program P

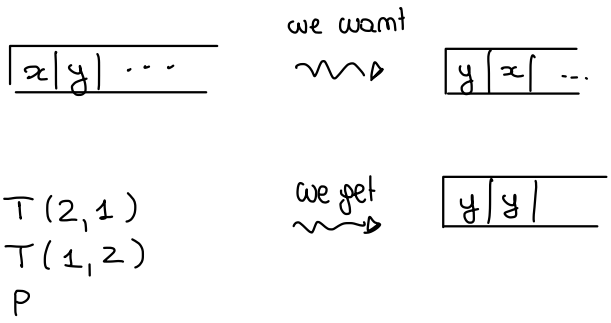
$$P[i_1, i_2, \dots, i_k \to i] \qquad (*)$$

program taking the input from $R_{i_1}, \dots, R_{i_k}$ and outputs in R_i without assuming that register different from input are $\emptyset$.

$T(i_1, 1)$
$\vdots$
$T(i_k, k)$
$Z(k+1)$
$\vdots$
$Z(p(P))$
P
$T(1, i)$

$(*)$

problem $P[2, 1 \to 1]$

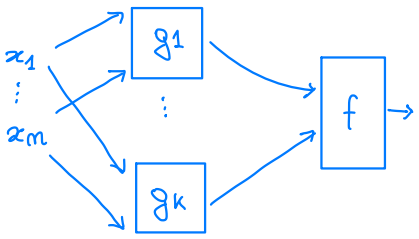


EXERCISE : Solve the problem !

* COMPOSITION

Given $f : \mathbb{N}^k \to \mathbb{N}$, $g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$

you define $h : \mathbb{N}^m \to \mathbb{N}$



$$h(\vec{x}) = \begin{cases} f(g_1(\vec{x}), \dots, g_k(\vec{x})) & \text{if } g_1(\vec{x})\downarrow, \dots, g_k(\vec{x})\downarrow \\ & f(g_1(\vec{x}), \dots, g_k(\vec{x}))\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

E.g. $z(x) = 0 \quad \forall x$

$\emptyset(x)\uparrow \quad \forall x$

$z(\emptyset(x))\uparrow \quad \forall x$

$U_1^2(x, y) = x$

$U_1^2(x, \emptyset(y))\uparrow \quad \forall x, y$

Proposition: $\mathcal{C}$ is closed under composition

proof

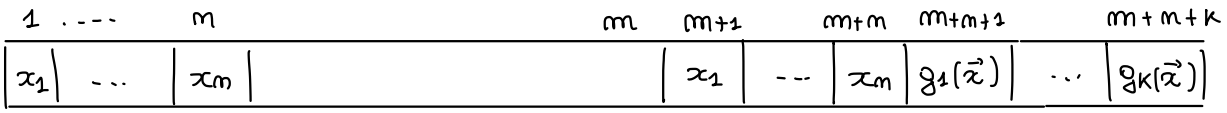
Given $f: \mathbb{N}^k \to \mathbb{N}$, $g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$ computable functions in $\mathcal{C}$

then

$h: \mathbb{N}^m \to \mathbb{N}$

$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$ is in $\mathcal{C}$

Let $F, G_1, \dots, G_k$ be programs (std form) for $f, g_1, \dots, g_k$



$$n = \max\{\rho(F), \rho(G_1), \dots, \rho(G_k), k, m\}$$

$T(1, m+1)$

$\vdots$

$T(m, m+m)$

$G_1\,[m+1 \dots m+m \to m+m+1]$

$\vdots$

$G_k\,[m+1 \dots m+m \to m+m+k]$

$F\,[m+m+1 \dots m+m+k \to 1]$

is the program for h, hence $h \in \mathcal{C}$ □

Example: $f(x_1, x_2) = x_1 + x_2$ is known to be in $\mathcal{C}$

$g(x_1, x_2, x_3) = x_1 + x_2 + x_3$

$g(x_1, x_2, x_3) = f(f(x_1, x_2), x_3)$

$= f(f(U_1^3(\vec{x}), U_2^3(\vec{x})), U_3^3(\vec{x}))$

$U_1^3(\vec{x})$ $U_2^3(\vec{x})$: $\mathbb{N}^3 \to \mathbb{N}$ $\mathbb{N}^3 \to \mathbb{N}$

$f(U_1^3(\vec{x}), U_2^3(\vec{x})) : \mathbb{N}^3 \to \mathbb{N}$

$U_3^3(\vec{x}) : \mathbb{N}^3 \to \mathbb{N}$

$f(f(U_1^3(\vec{x}), U_2^3(\vec{x})), U_3^3(\vec{x})) : \mathbb{N}^3 \to \mathbb{N}$

Example: let f be computable and total

$$Q_f(x,y) = \text{“}f(x) = y\text{”} \qquad \text{decidable ?}$$

We know

$$\chi_{Eq} : \mathbb{N}^2 \to \mathbb{N}$$

$$\chi_{Eq}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \text{computable}$$

hence Q_f is decidable since

$$\chi_{Q_f}(x,y) = \chi_{Eq}(f(x), y) \qquad \text{computable by composition}$$

* Primitive Recursion

$$\begin{cases} 0! = 1 \\ (m+1)! = m! * (m+1) \end{cases}$$

($m!$: known m)

$$\begin{cases} fib(0) = 1 \\ fib(1) = 1 \\ fib(m+2) = fib(m) + fib(m+1) \end{cases}$$

($fib(m)$, $fib(m+1)$: known)

Def. Given $f : \mathbb{N}^k \to \mathbb{N}$

$g : \mathbb{N}^{k+2} \to \mathbb{N}$

define $h : \mathbb{N}^{k+1} \to \mathbb{N}$

$$\begin{cases} h(\vec{x}, 0) = f(\vec{x}) \\ h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y)) \end{cases}$$

take $x \in \mathbb{R}$

$$x = \frac{\sqrt{x}}{e^x} \cdot \log x$$

- is there a solution?
- is it unique? canonical?

$(Y, \subseteq)$ complete partial order, continuous transformation, unique solutions $\longrightarrow$ existence of a (least) solution

\- - - - - - $\longrightarrow$ induction

Examples:

* $h: \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x+y$

$$\begin{cases} x+0 = x \\ x+(y+1) = (x+y)+1 \end{cases}$$

$f(x) = x = U_1^1(x)$

$g(x,y,z) = z+1$

* $h': \mathbb{N}^2 \to \mathbb{N}$

$h'(x,y) = x*y$

$$\begin{cases} x*0 = 0 \\ x*(y+1) = (x*y)+x \end{cases}$$

$f(x) = 0$

$g(x,y,z) = z+x$

Proposition: The class $\mathcal{C}$ is closed under primitive recursion

proof

Let $f: \mathbb{N}^k \to \mathbb{N}$, $g: \mathbb{N}^{k+2} \to \mathbb{N}$ be in $\mathcal{C}$

and let F, G programs (std form) for f and g

Define $h: \mathbb{N}^{k+1} \to \mathbb{N}$

$h(\vec{x},0) = f(\vec{x})$

$h(\vec{x},y+1) = g(\vec{x},y,h(\vec{x},y))$

Idea.

$h(\vec{x},0) = f(\vec{x})$ use F

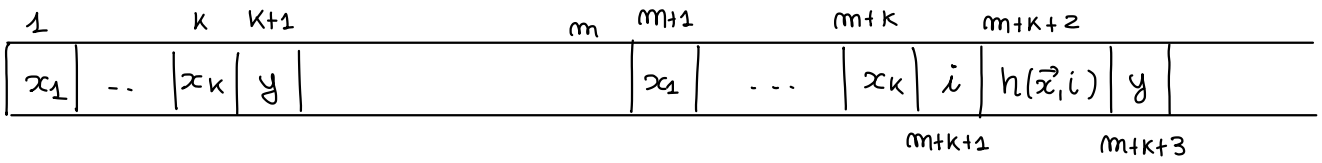
$h(\vec{x},1) = g(\vec{x},0,h(\vec{x},0))$ use G

$\vdots$

$h(\vec{x},i) = g(\vec{x},i-1,h(\vec{x},i-1))$ " "

$i = y$? if so → output

otherwise increment $i = i+1$ and continue



$$m = \max\{\rho(F), \rho(G), K+2\}$$

$T(1, m+1)$
:
$T(K, m+K)$

$T(K+1, m+K+3)$

$F[m+1, \ldots, m+K \to m+K+2]$ // $h(\vec{x},0) = f(\vec{x})$

LOOP: $J(m+K+1, m+K+3, RES)$ // $i = y$?

$G[m+1 \ldots m+K+2 \to m+K+2]$ // $h(\vec{x}, i+1) = g(\vec{x}, i, h(\vec{x},i))$

$S(m+K+1)$ // i++

$J(1,1,LOOP)$

RES: $T(m+K+2, 1)$

□