DECOMPOSITION (only with respect to $\mathcal{L}^n$ but true more in general)

thm of Lebesgue

μ Radon in $\mathbb{R}^n$

for a.e $x \in \mathbb{R}^n$ $\exists$ finite $\lim_{r \to 0^+} \frac{\mu(B(x,r))}{\omega_n r^n} = f_\mu(x)$

($\omega_n r^n \to \mathcal{L}^n(B(x,r))$)

and $f_\mu \in L^1_{loc}(\mathbb{R}^n)$

f_μ DENSITY of μ w.r.t. $\mathcal{L}^n$.

$\delta > 0$ U open $\exists$ COUNTABLE COLL. $B^n \subseteq U$ DISJOINT $\dim B_i \le \delta$ $|U \setminus \cup_n B_n| = 0$

PROOF BASED ON COVERING THM BY VITALI

Definition Let μ, ν Radon

$\mu \ll \nu$ μ is ABSOLUTELY CONT. w.r.t. ν if

$\forall A \subseteq \mathbb{R}^n$ with $\nu(A) = 0$ then $\mu(A) = 0$

$\mu \perp \nu$ μ and ν are MUTUALLY SINGULAR if

$\exists B \in \mathcal{B}$ $\mu(B) = 0$ $\nu(\mathbb{R}^n \setminus B) = 0$

Absolutely continuous measures.

1) THEOREM Let $\nu \ll \mathcal{L}^n$. Let f_ν the density of ν

then $\forall A \in \mathcal{B}$ $\quad \nu(A) = \int_A f_\nu \, dx$ $\quad$ (so $\int_{\mathbb{R}^n} g \, d\nu = \int_{\mathbb{R}^n} g f_\nu \, dx$)

(every absolutely cont. measure w.r.t. $\mathcal{L}^n$ can be written as the integral of its density w.r.t. $\mathcal{L}^n$)

2) Let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$ μ Radon $f \geq 0$

Define $\nu(A) := \int_A f \, d\mu$ $\forall A \in \mathcal{B}$ and $\nu(C) = \inf_{A \in \mathcal{B}} \{\nu(A), A \supseteq C\}$

Then ν is a RADON MEASURE $\nu \ll \mu$. Let $\mu = \mathcal{L}^n$ then $f = f_\nu$.

$\nu(\phi) = 0$ $\quad \forall K$ compact $\nu(K) = \int_K f \, d\mu < +\infty$

$\nu(\cup_i A_i) = \sum_i \nu(A_i)$ $\quad A_i \cap A_j = \phi$ by MONOTONE CONVERG.

(exercise)

$f \in L^1(\mathbb{R}^n, \mu) \iff \nu(\mathbb{R}^n) < +\infty$!

3) if $f \in L^1_{loc}(\mathbb{R}^n, \mu)$ $\quad f = f^+ - f^-$

$$\nu^+(A) = \int_A f^+ \, d\mu \qquad \nu^-(A) = \int_A f^- \, d\mu \qquad \nu^+ \perp \nu^-$$

we call $\nu = \nu^+ - \nu^-$ is a SIGNED RADON MEASURE

$|\nu| =$ TOTAL VARIATION MEASURE $= \nu^+ + \nu^-$

Ex δ_{x_0} $x_0 \in \mathbb{R}^n$ $\delta_{x_0} \perp \mathcal{L}$ $B = \{x_0\}$ $\mathcal{L}^n(\{x_0\}) = 0$ $\delta_{x_0}(\mathbb{R}^n \setminus \{x_0\}) = 0$

δ_{x_0} has no DENSITY $f \equiv 0$ a.e x

$$f_{\delta_{x_0}}(x) = \lim_{r \to 0^+} \frac{\delta_{x_0}(B(x,r))}{\omega_n r^n} = 0 \quad \forall x \neq x_0$$

since $x \neq x_0 \Rightarrow \forall r < |x - x_0|$

$\delta_{x_0}(B(x,r)) = 0$.

Theorem of DECOMPOSITION.

μ Radon measure. $\exists!\ \mu_{ac} \ll \mathcal{L}^n$ $\mu_s \perp \mathcal{L}^n$ such that

1) $\mu = \mu_{ac} + \mu_s$
2) $f_\mu = f_{\mu_{ac}}$ $\quad f_{\mu_s} = 0$
3) $\forall A \in \mathcal{B}$ $\quad \mu(A) = \int_A f_\mu(x)\,dx + \mu_s(A)$.

MEASURE-THEORETIC BDRY.

Lebesgue theorem μ Radon $\quad f \in L^1_{loc}(\mathbb{R}^n, \mu)$

for μ-a.e. $x \in \mathbb{R}^n$ $\quad \lim_{r \to 0^+} \underbrace{\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\,d\mu}_{\fint_{B(x,r)} f(y)\,d\mu} = f(x)$

$f \in L^p_{loc}(\mathbb{R}^n, \mu)$ $\quad$ for a.e. x $\quad \lim_{r \to 0^+} \fint_{B(x,r)} |f(y) - f(x)|^p\,d\mu = 0$.

Obs 1 $\mu = \mathcal{L}^n$ the canonical representative of $f \in L^1_{loc}(\mathbb{R}^n)$ is $f(x) = \lim_{r \to 0^+} \fint_{B(x,r)} f(y)\,dy$ (for a.e. x)

$f(x) = 0$ for x such that previous limit do not exists.

Let E be a MEASURABLE set (by Lebesgue) $E \subseteq \mathbb{R}^n$

$\downarrow$ then $\chi_E \in L^1_{loc}(\mathbb{R}^n)$ (note that if $|E| < +\infty$ $\chi_E \in L^1(\mathbb{R}^n)$)

for a.e $x \in E$ $\chi_E(x) = 1 = \lim_{r \to 0^+} \dfrac{|E \cap B(x,r)|}{\omega_n r^n}$ [NOTE THAT $|E \cap B(x,r)| = \int_{B(x,r)} \chi_E(y)\,dy$]

$\{x \in E$ with $\lim_{r \to 0^+} \dfrac{|E \cap B(x,r)|}{\omega_n r^n} = 1\}$ are points of DENSITY ONE

[MEASURE THEORETIC INTERIOR of E.

NOTE that if E is an open set $\underline{\forall x \in E}$ $\lim_{r \to 0^+} \dfrac{|E \cap B(x,r)|}{\omega_n r^n}$

(measure theoretic interior $= E$ if E is OPEN)

for a.e. $x \in \mathbb{R}^n \setminus E$ $\quad 0 = \chi_E(x) = \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n}$

↓ measure theoretic exterior of $E = \{ x \in \mathbb{R}^n \setminus E$ where $\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} = 0 \}$

MEASURE THEORETIC BOUNDARY of E = $\partial_* E$

$= \{ x \in \mathbb{R}^n$ such that either $\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n}$ DOES NOT EXIST

or $\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} \in (0,1) \}$ OBSERVE that $|\partial_* E| = 0$ by its definition

ex E open set of class C^1

measure theoretic bdry of E = ∂E (topological bdry)

$= \{ x \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} = \frac{1}{2} \}$

$\forall x \in \mathbb{R}^n$ DENSITY of x in E if it EXISTS $= \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n}$.

Ex

Let q_i be a enumeration of $\mathbb{Q}$

$$E = \bigcup_{i=0}^{\infty} \left(-\frac{1}{2^i} + q_i,\ q_i + \frac{1}{2^i}\right)$$

$\downarrow$

E is OPEN

$$\bar{E} = \mathbb{R}$$

$$|E| \le \sum_{0}^{+\infty} \left(\frac{1}{2^i}\right) \frac{1}{2} = \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}}\right] = 1$$

Topological bdry $\bar{E} \setminus E$ ($|\bar{E} \setminus E| = +\infty$)

Measure theoretic bdry $|\partial_* E| = 0$

Note that $\bar{E} \setminus E = \mathbb{R} \setminus E$ so there is NO EXTERIOR of E

since $\partial_* E \subsetneq \bar{E} \setminus E$ and $|\bar{E} \setminus E| = +\infty$ $|\partial_* E| = 0$.

$\downarrow$

for a.e $x \in \mathbb{R} \setminus E$ density is 0

RIESZ DUALITY THEOREMS

① (X,τ) TOPOLOGICAL SPACE, X linear space $\quad X^* = \{$ linear continuous functionals $T: X \to \mathbb{R}\}$

T linear if $T(ax+by) = aTx + bTy \quad a,b \in \mathbb{R} \quad x,y \in X$

T (seq.) continuous if $x_n \to x$ in $X \Rightarrow Tx_n \to Tx$

② $(X, \|\cdot\|_X)$ Banach space $\quad X^*$ Banach with the norm

$$\|T\| = \sup_{\|x\| \le 1} |Tx|$$

RIESZ THEOREM $\quad (L^p(U))^* = L^q(U) \quad \forall p \in [1,+\infty) \quad \left(\frac{1}{p}+\frac{1}{q}=1\right)$

(q: conjugate exponent)

means $\forall T: L^p(U) \to \mathbb{R}$ linear continuous $\exists g \in L^q(U)$ with

$Tf = \int_U f g \, dx \quad \|T\| = \|g\|_{L^q}$ (note that by Hölder $\forall g \in L^q(U)$ $f \mapsto \int_U f g \, dx$ is in $(L^p(U))^*$)

L^p $p \in (1, +\infty)$ is REFLEXIVE $(L^p)^{**} \simeq L^p$

whereas L^1 NOT $(L^\infty)^* \supsetneq L^1$

DUAL of L^∞ is very large, not useful but we want to find

$L^1 \subsetneq (L^\infty)^*$ $\quad x_0 \in U$ U open set

Let $T : (C_0(U), \|\cdot\|_\infty) \to \mathbb{R}$ $\quad$ T is a linear continuous functional

$\phi \longmapsto \phi(x_0)$

T can be extended to $T : (L^\infty, \|\cdot\|_\infty) \to \mathbb{R}$ by HAHN BANACH

and $\exists\, f \in L^1(U)$ such that $T(\varphi) = \varphi(x_0) = \int_U \phi f \, dx \quad \forall \phi \in C_0(U)$

assume by CONTRADICTION it exists

Let $\phi_n(x) = \rho((x - x_0)n)$ $\qquad \rho(y) = \begin{cases} c\, e^{\frac{1}{|y|^2 - 1}} & |y| < 1 \\ 0 & |y| \geq 1 \end{cases}$

$\phi_n(x) \to 0$ a.e.

$|\phi_n(x) f| \leq C \Rightarrow \int_U \phi_n f \, dx \to 0$ by Lebesgue / whereas $\phi_n(x_0) = \rho(0) = c e^{-1}$ $\forall n$.

SO BY REACH AN ABSURD

L^p $p \in (1, +\infty)$ is REFLEXIVE $(L^p)^{**} \simeq L^p$

whereas L^1 NOT $(L^\infty)^* \supsetneq L^1$

DUAL of L^∞ is very large, not useful but we want to find

Moreover L^p $p \geq 1$ are all separable spaces ($\exists$ countable dense subset, take simple functions with rational coeff. and with A_i which are rectangles with rational coeff.)

whereas L^∞ is NOT SEPARABLE ($\nexists$ COUNTABLE FAMILY $f_n \in L^\infty$ WHICH IS DENSE)

$\forall x \in \mathbb{R}^n \quad f_x = \chi_{B(x,1)} \quad \|f_x - f_y\|_\infty = 1 \quad x \neq y$

So $L^p(U)$ are reflexive for $p \in (1, +\infty)$

separable for $p \in [1, +\infty)$

$(L^1)^* = L^\infty$. L^∞ is the dual of a reflexive space.

WHY WE ARE INTERESTED IN REFLEXIVITY / SEPARABILITY of the PREDUAL?

Theorem $(X, \|\cdot\|_X)$ then $\dim X = +\infty \iff B = \{\|x\| \le 1\}$ is NOT COMPACT.

KAKUTANI If $(X, \|\cdot\|_X)$ is reflexive then $B(0,1)$ is weakly (sequentially) compact

i.e. $\forall x_n \in X$ $\|x_n\|_X$ $\exists\, x_{n_k}, x \in X$ s.t. $T x_{n_k} \to T x$ $\forall T \in X^*$

$\|x\| \le \liminf_k \|x_{n_k}\|$

$\forall f_n \in L^p(U)$ $p \in (1, +\infty)$ $\|f_n\| \le C$ $\exists\, f_{n_k}, f \in L^p(U)$ s.t.

$$\int_U f_{n_k} g \, dx \to \int_U f g \, dx \quad \forall g \in L^q(U) \qquad (f_{n_k} \to f)$$

$$\|f\|_p \le \liminf_k \|f_{n_k}\|_p$$

BANACH – ALAOGLU If $(X, \|\cdot\|_X)$ is the DUAL of a separable space then $B(0,1)$ is weak* sequentially compact

$(\|x_m\| \leq C \Rightarrow \exists x_{m_k}, x \in X$ such that $\forall y \in Y \quad Y^* \simeq X$

$x_m(y) \to x(y)$ (recall that $X = Y^*$ so x, x_m are linear continuous functional on Y)

Also true when X is a Frechet space and the dual of a separable space.

↓

$f_m \in L^\infty \quad \|f_m\|_\infty \leq C \Rightarrow \exists f_{m_k}, f \in L^\infty$ such that

$\forall g \in L^1(U) \quad \int_U f_{m_k} g \, dx \to \int_U f g \, dx \quad f_{m_k} \overset{*}{\rightharpoonup} f$