INTEGRATION on SPHERES (special case of CO-AREA formula ch. 2.7 Folland)

$\partial B(0,1) = \{x \in \mathbb{R}^n,\ |x| = 1\}$ sphere $\quad \partial B(0,r) = \{x \in \mathbb{R}^n,\ |x| = r\}$

$\sigma(\partial B(0,1))$ = area of the sphere $= \int_{\partial B(0,1)} dS$

"POLAR COORDINATES"

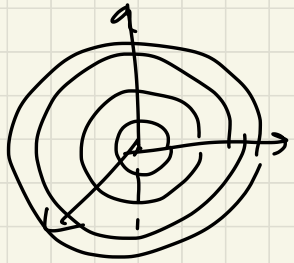
$\phi : x \in \mathbb{R}^n \setminus \{0\} \longrightarrow (r, x') \in [0, +\infty) \times \partial B(0,1)$ Continuous bijection

$r = |x| \qquad x' = \frac{x}{|x|} \quad x \neq 0 \qquad x = r x'$

$f : \mathbb{R}^n \to \mathbb{R}$ measurable $\quad (f \geq 0$ or $f \in L^1(\mathbb{R}^n))$

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^{+\infty} r^{n-1} \int_{\partial B(0,1)} f(r x')\,dS(x')\,dr$$

$$\left(= \int_0^{+\infty} \int_{\partial B(0,r)} f(y)\,dS(y)\,dr \right)$$



Obs if $f(x) = \phi(|x|)$ (that is: f has radial symm.)

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^{+\infty} r^{n-1}\,\phi(r)\,\mathcal{A}(\partial B(0,1))\,dr$$

Ex $f(x) = \dfrac{1}{|x|^\alpha}$

1) $f \cdot \chi_{B(0,R)} \in L^p(\mathbb{R}^n) \iff \alpha \cdot p < n$

$$\int_{\mathbb{R}^n} |f\chi_{B(0,R)}|^p\,dx = \int_0^{+\infty} r^{n-1} \int_{\partial B(0,1)} \frac{1}{r^{\alpha p}}\,\chi_{B(0,R)}(rx')\,dS(x') =$$

$$= \int_0^R r^{n-1}\,\frac{1}{r^{\alpha p}}\,dr < +\infty \iff \alpha p < n$$

2) $f\,\chi_{\mathbb{R}^n \setminus B(0,R)} \in L^p(\mathbb{R}^n) \iff \alpha p > n$ (exercise)

COMPUTATION of $\mathcal{L}(\partial B(0,1))$ and $|B(0,1)|$ = volume of $B(0,1)$

$\omega_n = |B(0,1)| = \int_{\mathbb{R}^n} \chi_{B(0,1)}\, dx =$ $\qquad |B(0,r)| = r^n \omega_n$

$\mathcal{L}(\partial B(0,1)) = \int_{\partial B(0,1)} dS$ $\qquad \mathcal{L}(\partial B(0,r)) = r^{n-1}\, \mathcal{L}(\partial B(0,1))$

$$\omega_n = \int_{\mathbb{R}^n} \chi_{B(0,1)}\, dx = \int_0^1 r^{n-1}\, \mathcal{L}(\partial B(0,1))\, dr = \frac{\mathcal{L}(\partial B(0,1))}{n} \Rightarrow \mathcal{L}(\partial B(0,r)) = n r^{n-1} \omega_n$$

$$|B(0,r)| = r^n \omega_n$$

integration on sphere

$$\pi^{n/2} = \int_{\mathbb{R}^n} e^{-|x|^2}\, dx \overset{\text{integration on spheres}}{=} \int_0^{+\infty} e^{-r^2} r^{n-1}\, \mathcal{L}(S^n)\, dr =$$

by symmetry & reduction to $n=2$

$$= \mathcal{L}(S^n)\, \frac{1}{2} \int_0^{+\infty} e^{-s}\, s^{\frac{n}{2}-1}\, ds = n\omega_n \frac{1}{2} \Gamma\left(\frac{n}{2}\right)$$

where Γ is the Gamma-function (by Euler)

$$\Gamma(t) = \int_0^{+\infty} e^{-s}\, s^{t-1}\, ds \qquad \left[\Gamma(t+1) = t\,\Gamma(t) \quad t>0 \right.$$
$$\left. \Gamma(1) = \Gamma(2) = 1 \right] \qquad \Gamma(n) = (n-1)!$$

$\lim_{t \to 0} \Gamma(t) = +\infty$

$$\omega_n = \frac{2\pi^{n/2}}{n\,\Gamma(\frac{n}{2})} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \qquad n\omega_n = \mathcal{H}(\partial B(0,1)) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

$\sim \cdot \sim$

GEOMETRIC MEASURE THEORY (Evans-Gariepy ch 1)

Def OUTER MEASURE (in E.G just measure)

X set $\quad \mu : \mathcal{P}(X) \to [0, +\infty]$ $\quad \mu(\phi) = 0$

$$\mu(A) \le \sum_i \mu(A_i) \quad \text{if } A \subseteq \bigcup_i A_i$$

ex: $X = \mathbb{R}$ $\quad \mathcal{L}^1(A) = \inf \{ \sum_i (\operatorname{diam} C_i) \;\; A \subseteq \bigcup_i C_i \}$

$C_i \subseteq \mathbb{R}$

$\operatorname{diam} C_i = \sup_{x,y \in C_i} |x-y|$

Def A is μ-measurable if $\forall B \in \mathcal{P}(X)$ (B "test")

$$\mu(B) = \mu(A \cap B) + \mu(B \setminus A)$$

(A allows to cut every test B in 2 sets $\begin{matrix} A\cap B \\ B\setminus A \end{matrix}$ in a "clean" way)

obv. $X, \emptyset$ are μ-meas. $\forall$ μ outer measure ($\mu(\emptyset) = 0$)

Prop A meas $\Rightarrow X \setminus A$ meas.
A_i meas $\Rightarrow \bigcup_i A_i$ meas.

$\Sigma_\mu = \{\mu\text{-measurable sets}\}$ is a σ-algebra (contains $\emptyset, X$, closed by complement and count. union)

Observation if A_i are measurables and $A_i \cap A_j = \emptyset$ $i \neq j$

then $\mu(\bigcup_i A_i) = \sum_i^{\infty} \mu(A_i)$

Proof $B_j := \bigcup_{i=1}^{j} A_i$ A_{j+1} is measurable

$$\Rightarrow \mu(B_{j+1}) = \mu(B_{j+1} \cap A_{j+1}) + \mu(B_{j+1} \setminus A_{j+1}) =$$
$$= \mu(A_{j+1}) + \mu(B_j).$$

$\mu_{|\mathcal{E}_\mu}$ μ restricted to $\mathcal{E}_\mu$ is a MEASURE (σ-additivity!)

↓

$A_i \cap A_j = \emptyset$ $\quad \sum_i \mu(A_i) = \mu(\cup_i A_i)$

$A_1 \subset A_2 \subset \dots$ $\quad \mu(\cup_i A_i) = \lim_{i \to +\infty} \mu(A_i)$

$A_1 \supset A_2 \supset \dots$ $\quad \mu(A_1) \neq +\infty$ $\quad \mu(\cap_i A_i) = \lim_i \mu(A_i)$

Def We say that μ is a Borel measure in X TOPOLOGICAL SPACE if $\mathcal{E}_\mu$ contains all open sets of X (so also closed...)

From now on $X = \mathbb{R}^n$ $\quad \mathcal{B}$ = Borel σ-algebra (generated by all open sets)

μ is Borel measure if $\mathcal{B} \subseteq \mathcal{E}_\mu$.

CARATHEODORY CRITERIUM: if $\mu(A \cup B) = \mu(A) + \mu(B)$ $\forall A, B \subseteq X$ $\text{dist}(A,B) > 0$ then closed sets are measurable $\Rightarrow \mathcal{B} \subseteq \mathcal{E}_\mu$.

μ is regular if $\forall A \in \mathcal{P}(X) \quad \exists B \in \Sigma_\mu \quad A \subseteq B, \ \mu(A) = \mu(B)$

μ is Borel regular if μ Borel and $\forall A \in \mathcal{P}(X)$

$\exists B \in \mathcal{B} \subseteq \Sigma_\mu \quad A \subseteq B \quad \mu(A) = \mu(B)$

μ is RADON if μ is Borel regular and locally finite ($\forall K \subseteq X \quad K$ compact $\quad \mu(K) < +\infty$)

Prop μ Radon on $\mathbb{R}^n$ Then

$\forall A \subseteq \mathbb{R}^n \quad \mu(A) = \inf \{ \mu(U), \ U \text{ open set} \quad A \subseteq U \}$

$\forall A \in \Sigma_\mu \quad \mu(A) = \sup \{ \mu(K), \ K \text{ compact} \quad K \subseteq A \}$

Def $f: X \to Y$ <u>μ measurable</u> $X \subseteq \mathbb{R}^n$ $Y \subseteq \mathbb{R}^m$

if for every open set $B \subseteq Y$ then $f^{-1}(B) \in \Sigma_\mu$

$f: X \to Y$ <u>Borel measurable</u> if $f^{-1}(B) \in \mathcal{B}_X$

(ex: continuous functions are Borel measurable)

INTEGRALS μ Radon measure on $\mathbb{R}^n$ $f: \mathbb{R}^n \to \mathbb{R}$ μ-meas.

$f^+ = \max(f, 0)$ $f^- = \max(-f, 0)$ $f = f^+ - f^-$ $|f| = f^+ + f^-$

g simple $g = \sum_1^\infty a_i \chi_{A_i}$ $a_i \geq 0$ $A_i \in \Sigma_\mu$ $\int_{\mathbb{R}^n} g \, d\mu = \sum_i a_i \mu(A_i)$

$$\int_{\mathbb{R}^n} f^+ \, d\mu = \inf \left[\int_{\mathbb{R}^n} g \, d\mu \quad g \geq f \ \mu\text{-a.e.} \quad g = \sum_1^\infty a_i \chi_{A_i} \right\}$$

(A_i disjoint)

$\int_{\mathbb{R}^n} f^+ \, d\mu < +\infty$ and $\int_{\mathbb{R}^n} f^- \, d\mu < +\infty \iff f \in L^1(\mathbb{R}^n, \mu)$

ALL THEOREMS (Fatou, dominated convergence, monotone conv. HOLDS.)

Ex Let $\mu = \delta_{x_0}$ $\quad \delta_{x_0}(A) = \begin{cases} 0 & x_0 \notin A \\ 1 & x_0 \in A \end{cases}$ $\quad \delta_0$ is Radon.

$$\int_{\mathbb{R}^n} f \, d\mu = f(x_0) \qquad \mu = \sum_{\substack{i \\ \text{in } \mathbb{R}}}^{+\infty} \delta_{i} \qquad \int_{\mathbb{R}} f \, d\mu = \sum_{k=0}^{+\infty} f(k)$$

PRODUCT MEASURES μ meas. on $\mathbb{R}^n$ $\quad \nu$ measure on $\mathbb{R}^k$

$\mu \times \nu$ measure on $\mathbb{R}^n \times \mathbb{R}^k$

$$\mu \times \nu (S) := \inf \left[\sum_i \mu(A_i) \nu(B_i) \qquad S \subseteq \bigcup_i (A_i \times B_i) \right]$$

Prop. 1) $A \in \Sigma_\mu$ $B \in \Sigma_\nu \Rightarrow (\mu \times \nu)(A \times B) = \mu(A) \nu(B)$

2) μ Radon, ν Radon, $S \subseteq \mathbb{R}^n \times \mathbb{R}^k$

for μ-a.e x $\quad S_x = \{ y \in \mathbb{R}^k \ (x,y) \in S \} \in \Sigma_\nu$

for ν-a.e y $\quad S_y = \{ x \in \mathbb{R}^n \ (x,y) \in S \} \in \Sigma_\mu$

$$(\mu \times \nu)(S) = \int_{\mathbb{R}^n} \nu(S_x) \, d\mu = \int_{\mathbb{R}^k} \mu(S_y) \, d\nu$$

Ex $\mathcal{L}^n = \mathcal{L}^1 \times \mathcal{L}^1 \times \dots \times \mathcal{L}^1$ (Lebesgue)