PRELIMINARIES FROM CALCULUS

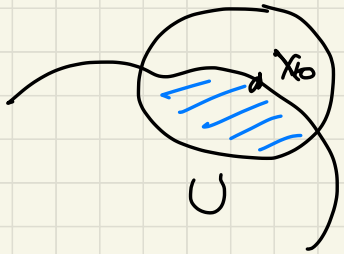
Def $U \subseteq \mathbb{R}^n$ open is of class C^k (or has boundary of class C^k) if

$\forall x_0 \in \partial U \ \exists r = r_{x_0} > 0$ and $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ of class C^k s.t.

(up to a rotation and translation of variables)

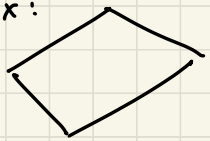
$$B(x_0, r) \cap U = \{ (x_1, \dots x_n) \text{ s.t. } x_n < \gamma(x_1, \dots x_{n-1}) \}$$

this implies $\partial U \cap B(x_0, r) = \{ x_n = \gamma(x_1, \dots x_{n-1}) \}$ but it is not sufficient!



U is analytic if $\gamma \in C^\omega(\mathbb{R}^{n-1})$

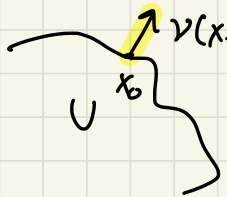
U is lipschitz if $\gamma \in C^{0,1}(\mathbb{R}^{n-1})$ ex:



U is Hölder if $\gamma \in C^{0,\alpha}(\mathbb{R}^{n-1})$

If U is of class C^1, $\forall x_0 \in \partial U$ $\exists\, \nu(x_0) \in \mathbb{R}^n$ $|\nu(x_0)| = 1$

EXTERIOR NORMAL TO U at x_0

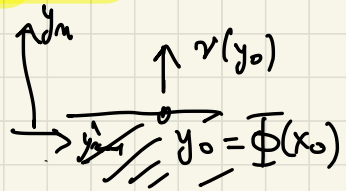


$$\nu(x_0) = (-\nabla\gamma(x_1^0 \dots x_{n-1}^0), 1)$$

- A typical procedure "STRAIGHTENING THE BOUNDARY".



$B(x_0,r) \cap U = \{ \gamma(x_1 \dots x_{n-1}) > x_n \}$ $\xrightarrow{\Phi}$



$$\Phi : B(x_0,r) \longrightarrow \mathbb{R}^n$$
$$(x_1 \dots x_n) \longmapsto (x_1, \dots x_{n-1}, x_n - \gamma(x_1 \dots x_{n-1})) = (y_1 \dots y_n)$$

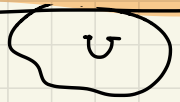
Φ is a C^1 diffeomorphism

$\det D\Phi(x) = 1$

(the bdy is locally flattened out)

Is it POSSIBLE to extend $v(x)$ to a FUNCTION $(\mathbb{R}^n \to \mathbb{R}^n)$ IN a NEIGHBORHOOD of the BOUNDARY?

$U \subseteq \mathbb{R}^n$ open bold



(GLOBAL!)

$d_S(x)$ = signed distance from U $= d(x, \bar{U}) - d(x, \mathbb{R}^n \setminus U)$

$$d_S(x) = \begin{cases} d(x, \partial U) & x \in \mathbb{R}^n \setminus U \\ -d(x, \partial U) & x \in U \end{cases}$$

"$\nabla d_S(x) = \nu(x)$" $x \in \partial U$

d_S is a 1-lipschitz continuous functions

$\delta > 0$ → TUBULAR NEIGHBORHOOD of ∂U

$$(\partial U)_\delta = \{ x, \ d_S(x) \in (-\delta, \delta) \} =$$
$$= \{ x, \ |\text{dist}(x, \partial U)| < \delta \}$$



Then (by IMPLICIT FUNCTION THM) If U bold of class C^k, $k \geq 2$

$\exists \delta > 0$ s.t. that $d_S \in C^k((\partial U)_\delta)$ $\nabla d_S = \nu \ \forall x \in \partial U$

$\nabla d_S \in C^{k-1}((\partial U)_\delta)$

$k=1$?

Federer (50) Possible $\Leftrightarrow$ ∂U has POSITIVE REACH

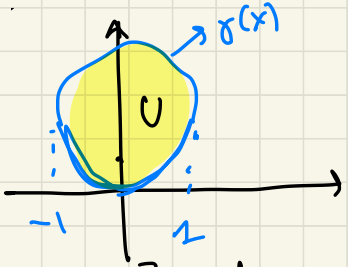
[that is $\exists r>0$ such that $\forall x \in \mathbb{R}^n$ $|d_S(x)|<r$

$\exists!\ \bar{x} \in \partial U$ such that $d(x,\partial U) = |x-\bar{x}|$

Note that U of class C^1 $\not\Rightarrow$ ∂U has positive reach!

$U = \{(x,y) \in \mathbb{R}^2 \quad y > |x|^{2-\varepsilon} \quad x \in [-1,1]\}$ $\varepsilon \in (0,1)$

$y \not< \gamma(x)$ ...



U bold of class C^1

For all $a>0$ sufficiently small $\rightarrow \exists\, x_1 = x_1(a) > 0$

$$d((0,a),\partial U)^2 = |(0,a) - (x_1, x_1^{2-\varepsilon})|^2 = |(0,a) - (-x_1, x_1^{2-\varepsilon})|^2 < |a|^2 = d((0,a),(0,0))^2$$

∂U HAS NOT POSITIVE REACH

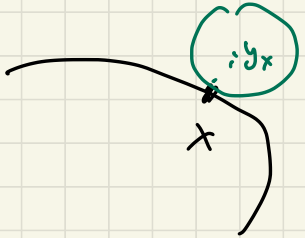
A SUFFICIENT (NOT NECESSARY!) CONDITION FOR POSITIVE REACH IS THE UNIFORM EXTERIOR AND INTERIOR SPHERE CONDITION

Def: U has the (UNIFORM) EXTERIOR SPHERE CONDITION if

$\exists r>0$ such that $\forall x \in \partial U \quad \exists y_x \in \mathbb{R}^n \setminus U$

$B(y_x, r) \cap U = \emptyset$

$\overline{B(y_x, r)} \cap \overline{U} = \{x\}$



(ex: U convex set has the ext ball condition with $r \in (0, +\infty)$!)

Def U has the (UNIFORM) INTERIOR SPHERE CONDITION if $\exists r>0$ such that $\forall x \in \partial U \quad \exists y_x \in U$

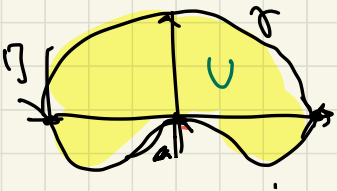
$B(y_x, r) \subseteq U \qquad \overline{B(y_x, r)} \cap \mathbb{R}^n \setminus U = \{x\}$



U has the UNIFORM EXTERIOR & INTERIOR SPHERE CONDITION $\Leftrightarrow$ U is of CLASS $\mathcal{C}^{1,1}$ ($\mathcal{C}^1$ functions with Lipschitz derivatives)

Ex : $U = \{(x,y) \in \mathbb{R}^2 \;/\; y > x^2 \lg|x| \text{ for } x \in [-1,1]\}$

+ : curve γ $y \in \gamma(x)$



U of class $\mathcal{C}^1$ but U in (0,0) has NOT THE EXTERIOR SPHERE CONDITION

(it is not possible to find $\delta > 0$ such that $\exists (0,y)$ $y<0$ with $B((0,y),\delta) \cap U = \emptyset$ $\overline{B((0,y),\delta)} \cap \overline{U} = \{(0,0)\}$)

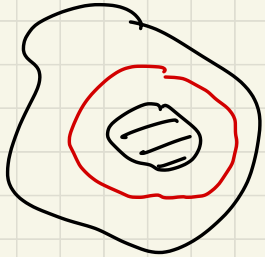
In (0,0) the "curvature" explodes
(the 2nd derivative of $x^2 \lg|x|$ ..)

PARTITIONS of UNITY (way to localize) [Folland ch 4 Prop. 4.4.1]

URYSHON KIND LEMMA

$U \subseteq \mathbb{R}^n$ open $K \subset\subset U$ compact

There $\exists \phi \in C_c^\infty(\mathbb{R}^n)$ $\begin{cases} \phi \equiv 1 \text{ on } K \\ \phi \equiv 0 \text{ on } \mathbb{R}^n \setminus U \\ 0 \le \phi(x) \le 1 \end{cases}$



proof Let ε_0 such that $\overline{K + B(0,3\varepsilon_0)} \subset\subset U$

$$f(x) = \begin{cases} 1 & x \in \overline{K + B(0,2\varepsilon_0)} \\ 0 & \text{elsewhere} \end{cases}$$

$f * \rho_{\varepsilon_0} \in C^\infty(\mathbb{R}^n)$ $\quad 0 \le f * \rho_{\varepsilon_0} \le \|f * \rho_{\varepsilon_0}\|_\infty \le \|f\|_\infty \|\rho_\varepsilon\|_1 = 1$

$\operatorname{supp} f * \rho_{\varepsilon_0} \subseteq \overline{K + B(0,2\varepsilon_0) + B(0,\varepsilon_0)} \subset\subset U \Rightarrow f * \rho_\varepsilon \equiv 0$ on $\mathbb{R}^n \setminus U$

$x \in K$ $\quad f * \rho_{\varepsilon_0}(x) = \int_{B(0,\varepsilon_0)} f(x-y)\, \rho_{\varepsilon_0}(y)\, dy = \int_{B(0,\varepsilon_0)} \rho_{\varepsilon_0}(y)\, dy = 1$.

$f(x-y) = 1$ since $x - y \in K + B(0,\varepsilon_0)$

Def $A \subseteq \mathbb{R}^n$ A partition of units in A is $\{\phi_\alpha\}_{\alpha \in I}$

1) $\phi_\alpha \in C^\infty(\mathbb{R}^n) \quad 0 \le \phi_\alpha \le 1$

2) $\forall x \in A \quad \exists I_x \subseteq I \quad \#I_x < +\infty$, U_x neighborhood of x
s.t. $\phi_\alpha(y) = 0 \quad \forall \alpha \notin I_x \quad y \in U_x$

3) $\forall x \in A \quad \sum_{\alpha \in I} \phi_\alpha(x) = 1$

The partition is SUBORDINATED to an open cover U_i of A if $\forall \alpha \ \exists i$ $\operatorname{supp} \phi_\alpha \subseteq U_i$.

If K is compact, with U_i open cover of K $(K \subseteq \bigcup_i U_i)$:
$\exists$ a partition of unit $\{\phi_j\}_{j \in \mathbb{N}}$ $\phi_j \in C^\infty_c(\mathbb{R}^n)$, with $\operatorname{supp} \phi_j \subseteq U_i$ $\exists i$

How to use ?

U bdd of class C^1. $\forall x_0 \in \partial U$ $\exists r_{x_0}$ $B(x_0, r_{x_0})$ open

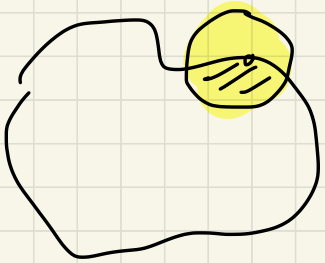
γ_{x_0} ..

$\rightarrow \partial U \subseteq \bigcup_{x_0 \in \partial U} B(x_0, r_{x_0})$ extract finite open cover (∂U is COMPACT)

and consider a partition of unit subordinated to such

open cover $\{\phi_j\}$?

$f \in L^1_{loc}$ $f\phi_j$ just see what happens in $\operatorname{supp}\phi_j \subseteq B(x_0^i, r_{x_0}^i)$



where ∂U graph of γ_{x_0} and U epigraph

In this way e.g. I may define dS

$$\int_{\partial U} f\, dS \qquad f = \sum_i f\phi_i \text{ on } \partial U$$

$$\int_{\partial U} f\, dS = \sum_j \int_{\partial U} f\phi_j\, dS = \sum_j \int_{\partial U \cap B(x_0^j, r_{x_0}^j)} f\phi_j\, dS \Rightarrow$$

$$\int_{\partial U \cap B(x_0^i, r_{x_0}^i)} f \phi_j \, dS = \int_{\partial U \cap B(x_0^i, r_{x_0}^i) \cap \mathbb{R}^{n-1}} f(x_1 \dots x_{n-1}, \gamma(x_1 .. x_{n-1})) \phi_j(..) \sqrt{1 + |\nabla \gamma_{x_i}|^2} \, dx_1 .. dx_{n-1}$$

we may then give a meaning to dS (the infinitesimal area term) by passing to parametrization