# DUBOIS REYMOND LEMMA

(Fundamental lemma calculus of variations)

$U \subseteq \mathbb{R}^n \quad f \in L^1_{loc}(U)$

$$\int_U f(x)\phi(x)dx = 0 \quad \forall \phi \in C_c^\infty(U) \iff f = 0 \text{ a.e. in } U$$

proof $\Leftarrow$ trivial

$\Rightarrow \quad V \subset\subset U \quad f \in L^1(\overline{V})$

$$g_V(x) = \begin{cases} \frac{f(x)}{|f(x)|} & x \in V \quad f(x) \neq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$g_V \in L^\infty_c(U) \qquad g_V * \rho_\varepsilon \in C^\infty_c(U) \quad$ for $\varepsilon \leq \varepsilon(V)$

By ass. $\int_U f(x)\, g_V * \rho_\varepsilon(x)\, dx = 0$

1) $\|g_V * \rho_\varepsilon\|_\infty \leq \|g_V\|_\infty \|\rho_\varepsilon\|_1 = 1$

2) fix $\varepsilon_0 < \varepsilon(V)$

$K_0 = \operatorname{supp}(g_V * \rho_{\varepsilon_0}) \subset\subset U$

$\forall \varepsilon \leq \varepsilon_0 \quad \operatorname{supp}(g_V * \rho_\varepsilon) \subseteq K_0$

$$0 = \int_U f(x)\,(g_V * \varrho_\varepsilon)(x) = \int_{K_0} f(x)\,[g_V * \varrho_\varepsilon](x)\,dx$$

$\varepsilon \le \varepsilon_0$

send $\varepsilon \to 0$ by Lebesgue

$$|f(x)|\,|g_V * \varrho_\varepsilon| \le |f(x)| \in L^1(K_0)$$

$g_V * \varrho_\varepsilon \to g_V$ in $L^1$ $\Rightarrow$ a.e up to a subsequence

$$f(g_V * \varrho_\varepsilon) \longrightarrow \begin{cases} |f| & \text{a.e in } V \\ 0 & \text{a.e in } K_0 \setminus V \end{cases}$$

$$0 = \int_{K_0} f(x)\,[g_V * \varrho_\varepsilon](x)\,dx \xrightarrow{\varepsilon \to 0} \int_V |f(x)|\,dx$$

$\Rightarrow f(x) = 0$ a.e in $V$ $\square$.

COROLLARY (in dim 1 — or more general $U$ CONNECTED)

$I$ interval $f \in L^1_{loc}(I)$

$\int_I f \phi' dx = 0 \ \forall \phi \in C^\infty_c(I) \iff \exists c \in \mathbb{R} \ f(x) = c$ a.e.

proof Fix $I = (a,b)$ $\Leftarrow$ trivial $\int_a^b c \cdot \phi'(x) dx = c[\phi(b) - \phi(a)] = 0$

$\Rightarrow$ Fix $\phi \in C^\infty_c(a,b)$, $h \in C^\infty_c(a,b)$ $\int_a^b h(x) dx = 1$, $h \geq 0$

$$\psi(x) = \int_a^x \varphi(y) dy - \int_a^b \phi(t) dt \int_a^x h(y) dy$$

$\psi(a) = \psi(b) = 0$ $\quad \psi'(x) = \phi(x) - \left[\int_a^b \phi(t) dt\right] h(x) \in C^\infty_c(a,b)$

$\to \psi \in C^\infty_c(a,b)$

$$0 = \int_a^b f(x) \psi'(x) dx = \int_a^b f(x) \phi(x) - \int_a^b f(x) h(x) \int_a^b \phi(t) dt \, dx =$$

$$= \int_a^b \phi(x) \left[ f(x) - \int_a^b f(t) h(t)\, dt \right] = 0 \qquad \forall \phi \in C_c^\infty(a,b)$$

$\Rightarrow$ DBR lemma $\quad f(x) - \int_a^b f(t) h(t) dt = 0$ for a.e $x$

$\Rightarrow f(x) = \underbrace{\int_a^b f(t) h(t) dt}_{\text{"c"}}$ for a.e $x \in (a,b)$.

A similar result also true in dim $n > 1$, for $U$ open CONNECTED set

Assume $\int_U f \cdot \frac{\partial \phi}{\partial x_i} dx = 0 \quad \forall \phi \in C_c^\infty(U)$ and $\forall i = 1 \dots n$

then $\exists\, c \in \mathbb{R}$ s.t. that $f \equiv c$ a.e.

(but more involved proof, using convolutions)

Back to COMPACT EMBEDDING of $C^1(\bar{U})$ into $C^0(\bar{U})$

with $U$ open bounded. Assume $U$ also CONNECTED (equiv. connected by ARCS) otherwise pass to a connected component.

Let $f_n \in C^1(U)$ with $\|f_n\|_{1,\infty} = \max\left\{\|f_n\|_\infty, \left\|\frac{\partial f_n}{\partial x_i}\right\|_\infty\right\} \le C$

then we may deduce $\|f_n\|_\infty \le C$ and

$\forall\, x, y \in \bar{U} \quad |f_n(x) - f_n(y)| \le C\, d_U(x,y)$

where $d_U(x,y)$ = geodesic distance in $U$ =

$$= \inf\left[ \int_0^T |\dot\gamma(t)|\,dt \;,\; \gamma:[0,T]\to\mathbb{R}^n \text{ rectifiable},\ \gamma(0)=x,\ \gamma(T)=y,\ \gamma(t)\in\bar{U}\ \forall t\in[0,T] \right]$$

In order to apply Ascoli Arzelà we need to deduce from $|f_m(x) - f_m(y)| \leq C\, d_U(x,y)$ that $f_m$ are equi-continuous (so we get that up to a subsequence $f_{m_k} \to f$ uniformly)

1) if $d_U(x,y) \leq K_U |x-y|$ TRUE for $U$ convex and for $U$ of lipschitz class (see in the following) (IN THIS CASE $f_m$ EQUILIPSCHITZ)

2) if $d_U(x,y) \leq K_U |x-y|^\alpha$ $\alpha < 1$ holds true for some $U$ of Holder class (IN THIS CASE $f_m$ EQUI-HÖLDER)

3) in general if $d_U(x,y)$ is continuous, I can deduce equicont of $f_m$.

BUT IN GENERAL CAN ALSO BE NOT CONTINUOUS

So, if $C^1(\bar{U})$ is defined as functions in $C^1(U)$ which can be continuously extended with their derivative up to the boundary

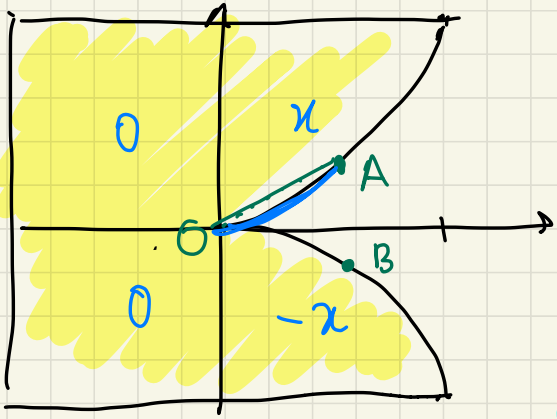
$C^1(\bar{U}) \longrightarrow C^0(\bar{U})$ is compact

if $d_U(x,y)$ is continuous.

(SO NEEDED GEOMETRIC CONDITIONS on $U$!)

(To overcome the problem one may use a more careful definition of $C^1(\bar{U})$)

Ex $U = (-1,1)^2 \setminus \{(x,y) \in \mathbb{R}^2 \quad 0 \le x \le 1 \quad -x^{3/2} \le y \le x^{3/2}\}$




IN THIS CASE CANNOT HOLD $d_U(x,y) \le C_U |x-y|$

$x \in (0,1)$ $\quad A = (x, x^{3/2})$ $\quad B = (x, -x^{3/2})$

$|A-B| = 2x^{3/2}$

$$d_U(A,B) \simeq 2\,|O-A| = 2\sqrt{x^2 + x^{9/4}} = 2x\sqrt{1+x^{1/4}}$$

$$\frac{d_U(A,B)}{|A-B|} \simeq \frac{\sqrt{1+x^{1/4}}}{x^{1/2}} \to +\infty \quad \text{as } x \to 0^+$$

GEODESIC DISTANCE NOT LIPSCHITZ

So $f(x,y) = \begin{cases} 0 & x \le 0 \\ x & x > 0, y > 0 \\ -x & x > 0 \ y < 0 \end{cases}$

$f(x, x^{3/2}) = x$

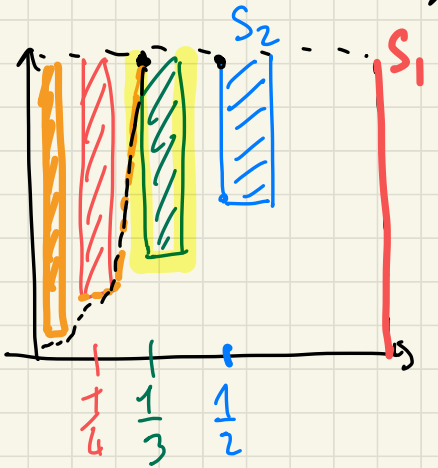
$f(x, -x^{3/2}) = -x$

$$|f(x,x^{3/2}) - f(x,-x^{3/2})| = 2x \qquad |A-B| = 2x^{3/2}$$

$f$ CANNOT BE LIPSCHITZ!

Ex geodesic distance NOT CONTINUOUS

$$U = (0,1)^2 \setminus \bigcup_{n=1}^{+\infty} S_n \qquad S_n = \left\{ \frac{1}{n} \le y \le 1 \quad \frac{1}{n} \le x \le \frac{1}{n} + \frac{1}{n^2} \right\}$$



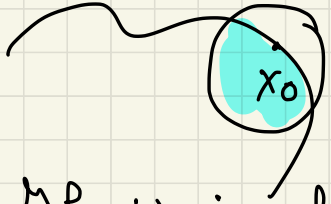

$$P_n = \left( \frac{1}{n} + \frac{1}{n^2}, 1 \right) \qquad Q_n = \left( \frac{1}{n}, 1 \right)$$

$$|P_n - Q_n| = \frac{1}{n^2} \longrightarrow 0$$

$$d_U(P_n, Q_n) \ge 2\left(1 - \frac{1}{n}\right) \not\longrightarrow 0$$

**Def**: $U \subseteq \mathbb{R}^n$ is an open set of class $C^k$ $(k \geq 0)$ if $\forall x_0 \in \partial U$ $\exists r_{x_0} > 0$ and $\gamma_{x_0} \in C^k(\mathbb{R}^{n-1})$ such that (up to a rotation/reordering of coordinates)

$$U \cap B(x_0, r) = \{ (x_1 .. x_n) \quad x_n < \gamma(x_1 .. x_{n-1}) \}$$




$$\partial U \cap B(x_0, r) = \{ x_n = \gamma(x_1 .. x_{n-1}) \}$$

If $U$ is of class $C^1$ at every $x \in \partial U$ we define the outer normal vector $\nu(x_0)$

$$\nu(x_0) = \frac{(-\nabla\gamma(x_1^0 .. x_{n-1}^0), 1)}{\sqrt{|\nabla\gamma|^2 + 1}} \qquad (|\nu(x_0)| = 1)$$

$U$ of lipschitz class if $\gamma$ just Lipschitz. $U$ Hölder if $\gamma$ Hölder..