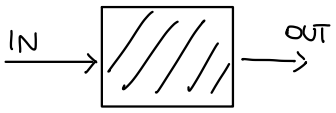
COMPUTABILITY (05/10/2026)

effective procedure (algorithm) $\Leftrightarrow$ computable function $\rightsquigarrow$ existence of non-computable functions

* Effective procedure

sequence of elementary steps

input data $\rightsquigarrow$ output data



deterministic

function f : {inputs} $\to$ {outputs}

(partial)

Def.: A function f is computable if there exists an algorithm such that the induced function is f.

* $GCD(x,y)$ = greatest common divisor (Euclid's algorithm)

* $f(n) = \begin{cases} 1 & n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$ (check if $2, \dots, n$ is a divisor)

* $h(n) = n^{th}$ digit in π

* $f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are } \underline{\text{exactly}} \ n \text{ consecutive digits } 5 \\ 0 & \text{otherwise} \end{cases}$

if $\pi = 3,145 \dots 7\ \underbrace{5555}_{\text{exactly 4 digits 5}}\ 4$

$f(4) = 1$

NOT A GOOD ALGORITHM !

idea: - compute all digits of π

- check whether there are exactly n digits 5 in a row

consider

$$g(m) = \begin{cases} 1 & \text{if } \pi \text{ includes at least } m \text{ digits } 5 \text{ in a row} \\ 0 & \text{otherwise} \end{cases}$$

if $\pi = 3{,}145 \ldots\ldots 7\,55554$

exactly 4 digits 5
(at least 3,2,1,0)

↓

$g(4) = 1$
$g(3) = 1$
$g(2) = 1$
$g(1) = 1$
$g(0) = 1$

more generally

$$g(m) = 1 \quad \Rightarrow \quad \forall\, m' \leq m \quad g(m') = 1$$

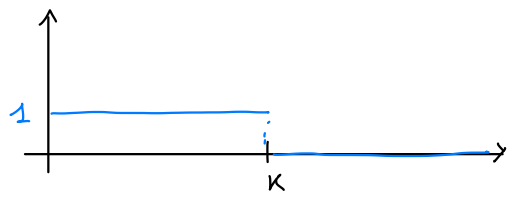
take

$$K = \sup \{ m \mid \text{there are } m \text{ digits } 5 \text{ in row in } \pi \}$$

two possibilities

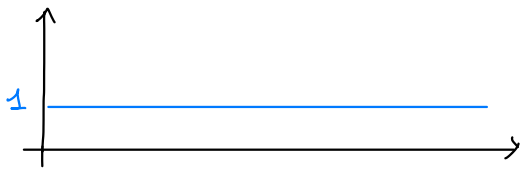
- K finite

$$g(m) = \begin{cases} 1 & \text{if } m \leq K \\ 0 & \text{otherwise} \end{cases}$$




```
function g(m):
    if m ≤ K:
        return 1
    else:
        return 0
```

- K is infinite $\quad g(m) = 1 \quad \forall m$




```
function g(m):
    return 1
```

Can't I play the same trick for arguing computability of f ?

$$f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are } \underline{\text{exactly}} \ n \text{ consecutive digits } 5 \\ 0 & \text{otherwise} \end{cases}$$

$$A = \{ n \mid \text{there are exactly } n \text{ digits } 5 \text{ in a row in } \pi \}$$

and define

```
function f(n):
    if n ∈ A:
        return 1
    else:
        return 0
```

↖ not an algorithm

A might be infinite

* Existence of non-computable functions

* Characteristics of a "good" algorithm

- finite length
- there is a computing agent which executes the algorithm
  - memory (unbounded)
  - discrete steps, deterministic, not probalistic
  - finite limit to the number and power of instructions
  - the computation can
    - → terminate after a finite number of steps → output
    - → diverge (never terminate) → no output

* Math notation

* $\mathbb{N} = \{0, 1, 2, \dots\}$

* A, B sets $\quad A \times B = \{(a,b) \mid a \in A, b \in B\}$

$A^m = A \times A \times \dots \times A$

ψ

$(a_1, a_2, \dots, a_m)$

* relations $\quad r \subseteq A \times B$

m divides n $\quad$ divides $= \{(m, m*k) \mid m, k \in \mathbb{N}\}$

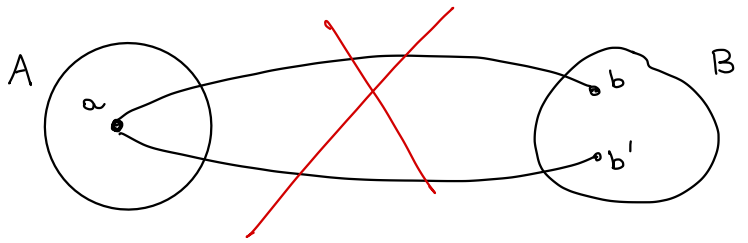
$\cap$

$\mathbb{N} \times \mathbb{N}$

* functions (partial) $\quad f : A \to B$

special relation $\quad f \subseteq A \times B$

$\forall a \in A \quad \forall b, b' \in B \quad (a,b), (a,b') \in f \Rightarrow b = b'$



$dom(f) = \{a \in A \mid \exists b.\ (a,b) \in f\}$

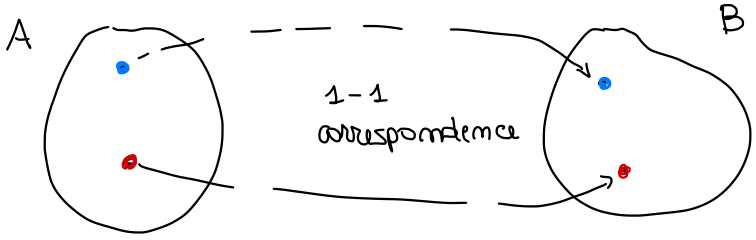
instead of $(a,b) \in f$ we write $f(a) = b$

if $a \in dom(f)$ $\quad f(a) \downarrow$

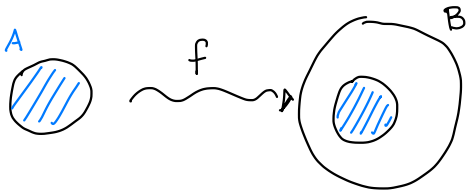
$a \notin dom(f)$ $\quad f(a) \uparrow$

* Cardinality

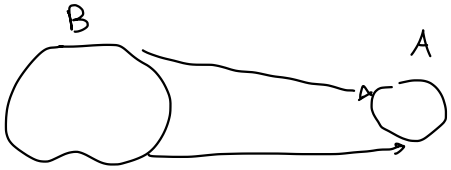
if A, B sets $\quad |A| = |B|$ if there is $f : A \to B$ bijective



$|A| \leq |B|$ if there is $f: A \to B$ injective



"equivalently" if there is a surjective function $g: B \to A$



* A countable (denumerable)

$|A| \leq |\mathbb{N}|$

surjective function $f: \mathbb{N} \to A$

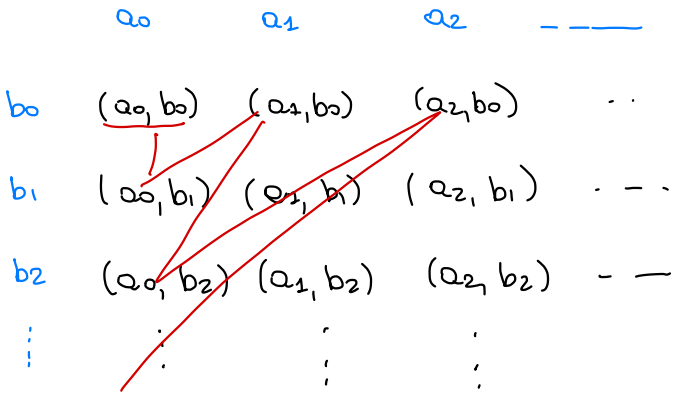
$f(0)$ $f(1)$ $f(2)$ ....

list of all elements of A

* A, B countable $\Rightarrow$ $A \times B$ countable

idea: A a_0 a_1 a_2 ---

B b_0 b_1 b_2 ---



* $A_0, A_1, \ldots, A_m, \ldots$ countable sets $\Rightarrow$ $\bigcup_{i \in \mathbb{N}} A_i$ is countable

* Existence of non computable function

we restrict to unary functions $f: \mathbb{N} \to \mathbb{N}$ (partial)

$\mathcal{F} = \{ f \mid f: \mathbb{N} \to \mathbb{N} \}$ set of all functions

Fix a model of computation $\rightsquigarrow$ set of algorithms $\mathcal{A}$

for each $A \in \mathcal{A}$ $\rightsquigarrow$ $f_A : \mathbb{N} \to \mathbb{N}$

Functions computable by the model

$\mathcal{F}_{\mathcal{A}} = \{ f \in \mathcal{F} \mid$ there exists $A \in \mathcal{A}$ such that $f_A = f \}$

$= \{ f_A \mid A \in \mathcal{A} \}$

Clearly $\mathcal{F}_{\mathcal{A}} \subsetneq \mathcal{F}$?

* An algorithm $A \in \mathcal{A}$ is a finite sequence of instructions from an instruction set I ← finite

$\mathcal{A} = I \cup I \times I \cup I \times I \times I \cup \dots$

$= \bigcup_{i \geq 1} I^i$

↑ countable union of countable sets $\Rightarrow$ countable

$|\mathcal{A}| \leq |\mathbb{N}|$

Now

$\mathcal{A} \longrightarrow \mathcal{F}_{\mathcal{A}}$ surjective

$A \longmapsto f_A$

$|\mathcal{F}_{\mathcal{A}}| \leq |\mathcal{A}| \leq |\mathbb{N}|$

* the set $\mathcal{G}$ of all functions is not countable

in fact, assume $|\mathcal{G}| \leq |\mathbb{N}|$

i.e. we can list the elements of $\mathcal{G}$

	f_0	f_1	f_2	-------
0	$f_0(0)$	$f_1(0)$	$f_2(0)$	...
1	$f_0(1)$	$f_1(1)$	$f_2(1)$	...
2	$f_0(2)$	$f_1(2)$	$f_2(2)$	...
⋮	⋮	⋮	⋮	

define $d: \mathbb{N} \to \mathbb{N}$ by changing systematically the diagonal

$$d(n) = \begin{cases} 0 & \text{if } f_n(n)\uparrow \\ \uparrow & \text{if } f_n(n)\downarrow \end{cases}$$

since $d \in \mathcal{G}$ it must be in the list, there is $n \in \mathbb{N}$ s.t. $d = f_n$

→ if $f_n(n)\downarrow \Rightarrow d(n)\uparrow \neq f_n(n)$

→ if $f_n(n)\uparrow \Rightarrow d(n) = 0 \neq f_n(n)$

contradiction

$\Rightarrow$ absurd

↳ $\mathcal{G}$ not countable $\Rightarrow$ $|\mathcal{G}| > |\mathbb{N}|$

* Summarising

$$\left.\begin{array}{l} \mathcal{G}_A \subseteq \mathcal{G} \\ |\mathcal{G}_A| \leq |\mathbb{N}| < |\mathcal{G}| \end{array}\right\} \Rightarrow \quad \boxed{\mathcal{G}_A \subsetneq \mathcal{G}}$$

* How many non-computable functions

$$|\mathcal{G} \setminus \mathcal{G}_A| > |\mathbb{N}|$$