Recap $(X, \|\cdot\|_X)$ $(Y, \|\cdot\|_Y)$ Banach spaces

$X \hookrightarrow Y$ cont. inclusion embedding if $X \subseteq Y$ as spaces

$$\exists C > 0 \quad \|x\|_Y \leq C \|x\|_X$$

$$(x_n \to x \text{ in } X \implies x_n \to x \text{ in } Y)$$

$X \hookrightarrow Y$ compact embedding if it is continuous and if $(x_n)_n$ is a bounded seq. in X $\|x_n\|_X \leq C$ then $(x_n)_n$ is a precompact sequence in Y

$(\exists\, x_{n_k},\ y \in Y \quad x_{n_k} \to y \text{ in } Y)$

$\sim \cdot \sim$

Ex $\left(C^1_{bdd}(\bar{U}), \|\cdot\|_{\infty,1}\right) \hookrightarrow \left(C(\bar{U}), \|\cdot\|_\infty\right)$ COMPACT EMBEDD.

$$\|f\|_{\infty,1} = \max_{|\alpha|\le 1} \|D^\alpha f\|_\infty = \max\left\{\|f\|_\infty, \left\|\frac{\partial f}{\partial x_i}\right\|_\infty\right\}$$

(the embedding is COMPACT DUE TO A.A.)

$$\|f_n\|_{\infty,1} \le C \Rightarrow \|f_n\|_\infty \le C \quad \text{and} \quad \left\|\frac{\partial f_n}{\partial x_i}\right\| \le C \ \forall i$$

f_n are equicontinuous]

$\Rightarrow$ conclude by A.A.

U bdd. $C^{0,\alpha}(U) = \{ f \in C(U) \quad \alpha\text{- Hölder continuous} \}$
$\subseteq C(\bar{U})$

on $C^{0,\alpha}(U)$ we put the norm

$$\|f\|_{0,\alpha} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

$(C^{0,\alpha}, \|\cdot\|_{0,\alpha})$ is a Banach space

$(C^{0,\alpha}(U), \|\cdot\|_{0,\alpha}) \hookrightarrow (C(\bar{U}), \|\cdot\|_\infty)$ is a COMPACT EMBEDDING (by Ascoli-Arzelà)

$\|f_n\|_{0,\alpha} \leq C \Rightarrow \|f\|_\infty \leq C$
$|f_n(x) - f_n(y)| \leq C|x-y|^\alpha \ \forall n$ $\Rightarrow \exists f \in C(\bar{U})$ ...

$0 < \beta < \alpha \leq 1$ $\left(C^{0,\beta}(U), \|\cdot\|_{0,\beta}\right) \hookrightarrow \left(C^{0,\alpha}(U), \|\cdot\|_{0,\alpha}\right)$

COMPACT EMBEDDING

Def. the SUPPORT of a function

$U \subseteq \mathbb{R}^n$ open $f: U \to \mathbb{R}$ measurable

$\operatorname{supp} f = U \setminus A$ $A = \bigcup \{ w \text{ open set } w \subseteq U \ f \equiv 0 \text{ in } w \}$

↓ relative closed in U.

(if f is continuous $\operatorname{supp} f = \overline{\{ x \in U, f(x) \neq 0 \}}$

$C_c(U) = \{ f: U \to \mathbb{R} \ f \in C(U), \operatorname{supp} f \text{ is compact in } U \} \subseteq C(\bar{U})$

$(C_c(U), \|\cdot\|_\infty)$ is not closed

$\overline{(C_c(U), \|\cdot\|_\infty)}^{\|\cdot\|_\infty} = (C_0(U), \|\cdot\|_\infty)$ Banach space

$C_0(U) = \{ f \in C(U) \quad "f = 0 \text{ on } \partial U" \} =$

$= \{ f \in C(U), \quad \forall \varepsilon > 0 \quad \{x \mid |f(x)| > \varepsilon\} \subset\subset U \}$

ex $f_n(x) = \begin{cases} e^{\frac{n-1}{n|x|^2 - n + 1}} & 0 \le |x| < \sqrt{1 - \frac{1}{n}} \\ 0 & \sqrt{1-\frac{1}{n}} \le |x| < 1 \end{cases}$

$f_n \in C_c(B(0,1))$ $\quad f_n \to f = e^{\frac{1}{|x|^2 - 1}}$ in $B(0,1)$ uniformly

"DUAL OF THE SPACE OF DISTRIBUTIONS"

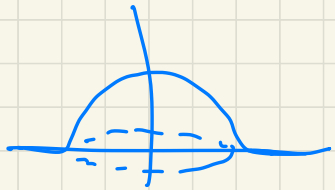
$C_c^\infty(U)$ $(= \mathcal{D}(U)) = \{ f \in C^\infty(U) \mid \operatorname{supp} f \text{ compact} \}$

$= \bigcap_{m \geq 0} C^m(U)$

SPACE of TEST FUNCTIONS

Obs
in $C_c^\infty(\mathbb{R}^n)$ there are NO ANALYTIC FUNCTIONS!

ex $f(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$ $f \in C_c^\infty(\mathbb{R}^n)$



$C_c^\infty(U)$ is not a Fréchet space! it is an INDUCTIVE LIMIT of Fréchet spaces

NON METRIZABLE

We put on $C_c^\infty(U)$ a topology τ such that $C_c^\infty(U)$ is a

HAUSDORFF LOCALLY COMPACT TOPOLOGICAL SPACE

$f_m \to f$ in $C_c^\infty(U) \iff$ $\exists K$ COMPACT $\subseteq U$ such that $\operatorname{supp} f_m, \operatorname{supp} f \subseteq K$

AND $\| D^\alpha f_m - D^\alpha f \|_\infty \to 0 \quad \forall \alpha$

$f \in L^p_{loc}(U) \Leftrightarrow f \in L^p(V) \ \forall V \subset\subset U$.

Recap. $U \subseteq \mathbb{R}^N$ open, $p \in [1, +\infty]$

$p \geq 1$ $L^p(U) = \{ f : U \to \mathbb{R} \ \text{measurable} \ \int_U |f(x)|^p dx < +\infty \}$

$\|f\|_p = \left(\int_U |f(x)|^p dx \right)^{1/p}$ $(L^p(U), \|\cdot\|_p)$ is a Banach space

$L^\infty(U) = \{ f : U \to \mathbb{R} \ \text{meas.}, \exists c \geq 0 \ |\{x \mid |f(x)| > c\}| = 0 \}$

$\|f\|_\infty = \operatorname{ess\,sup}_U |f(x)| = \inf \{ c : |\{x \mid |f(x)| > c\}| = 0 \}$

$(L^\infty(U), \|\cdot\|_\infty)$ is a Banach space.

NB we identify functions in $L^p(U)$ if they are a.e equal (that is $f \equiv g$ if $\{x \mid f(x) \neq g(x)\}$ has measure 0).

$(C_0(U), \|\cdot\|_\infty) \hookrightarrow (L^\infty(U), \|\cdot\|_\infty)$ continuous

$(C(\bar{U}), \|\cdot\|_\infty) \hookrightarrow (L^\infty(U), \|\cdot\|_\infty)$ continuous (U bdd)

Thm 1) $C_c^\infty(U)$ is DENSE in $L^p(U)$ $\forall\, p \in [1, +\infty)$
(with respect to $\|\cdot\|_p$) $\left(\forall f \in L^p(U) \;\; \exists\, \phi_n \in C_c^\infty(U) \;\; \|\phi_n - f\|_p \to 0 \;\; n \to +\infty\right)$

2) $\overline{C_c^\infty(U)}^{\|\cdot\|_\infty} = C_0(U) \subsetneq L^\infty(U)$

proof 2) by A.A. 1) by REGULARIZATION BY CONVOLUTION

PRODUCT of CONVOLUTION $f, g \in L^1_{loc}(\mathbb{R}^n)$

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \qquad \text{(if it exists)}$$

YOUNG INEQUALITY:
$f \in L^p(\mathbb{R}^n)$ $g \in L^q(\mathbb{R}^n)$ then
$f * g \in L^r(\mathbb{R}^n)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$
$\|f * g\|_r \leq \|f\|_p \|g\|_q$

(ex $f, g \in L^1 \Rightarrow f * g \in L^1$)

$f \in L^\infty,\ g \in L^1 \Rightarrow f * g \in L^\infty$ ...)

SUPPORT

$$\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$$

REGULARIZATION

$f \in L^1_{loc}(\mathbb{R}^n)$ $g \in C_c(\mathbb{R}^n)$

$f * g \in C(\mathbb{R}^n)$

$g \in C^k_c(\mathbb{R}^n)$

$f * g \in C^k(\mathbb{R}^n)$

$D^\alpha(f * g) = f * D^\alpha g$

Let $\rho(x) = \begin{cases} \bar{c}\, e^{\frac{1}{|x|^2 - 1}} & |x| < 1 \\ 0 & \text{elsewhere} \end{cases}$

$\bar{c}$ chosen s.t. $\int_{\mathbb{R}^n} \rho(x) = 1$

MOLLIFIERS

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right) \in C^\infty_c(\mathbb{R}^n)$$

$\operatorname{supp} \rho_\varepsilon = \overline{B(0, \varepsilon)}$

$\int_{\mathbb{R}^n} \rho_\varepsilon(x)\, dx = 1$

SKETCH of PROOF (1)

$f \in L^p(U)$ $\quad U$ open $\quad U = \bigcup_k V_k \quad V_k \subset V_{k+1} \quad V_k \subset\subset U$

$f_k = f \chi_{V_k} \in L^p(U)$ with compact support

$f_k \to f$ in $L^p(U)$ as $k \to +\infty$ $\quad (\|f_k - f\|_{L^p(U)} \to 0$ as $k \to +\infty)$

$\forall k \; \exists \varepsilon_k > 0$ s.t. that $f_k * \rho_\varepsilon \in C_c^\infty(U) \quad \forall \varepsilon < \varepsilon_k$

$(\operatorname{supp} f_k * \rho_\varepsilon \subseteq \overline{V_k} + \overline{B(0,\varepsilon)} \subset\subset U$ if ε is sufficiently small$)$

$$f_k * \rho_\varepsilon(x) - f_k(x) = \int_{B(0,\varepsilon)} f_k(x-y)\rho_\varepsilon(y)\,dy - \int_{B(0,\varepsilon)} f_k(x)\rho_\varepsilon(y)\,dy$$

$$\int_U |f_k * \rho_\varepsilon(x) - f_k(x)|^p dx \overset{\text{Jensen}}{\leq} \int_U \int_{B(0,\varepsilon)} |f_k(x-y) - f_k(x)|^p \rho_\varepsilon(y)\,dy$$

$\leq \dots$

(Recap **Jensen inequality**:

if $\Phi : \mathbb{R} \to \mathbb{R}$ is a CONVEX FUNCTION

and $g \geq 0$ $\int g(y)\,dy = 1$

then $$\Phi\left(\int f(y)\,g(y)\,dy\right) \leq \int \Phi(f(y))\,g(y)\,dy$$

(proof by using support hyperplane of convex f.)

Apply to $\Phi(t) = |t|^p$ convex

$g(y) = \rho_\varepsilon(y)$