PLAN of the COURSE

1) Preliminaries (calculus + measure theory)
   Radon measures (n'20), Hausdorff measures
2) theory of distributions (Schwartz '50)
3) Sobolev space (Sobolev '30)
4) geometric measure theory, BV functions

REF { EVANS-GARIEPY "Fine properties of functions"
FOLLAND, "Real Analysis"
EVANS "PDE" AMS

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All the information at the MOODLE PAGE

Feel free to ask questions (directly, by email, or to ask for our office-hour).

PRELIMINARIES. SPACE of TEST FUNCTIONS

$U \subseteq \mathbb{R}^n$ U open set (it could also be $\mathbb{R}^n$) $V \subset\subset U$ if $\overline{V}$ is compact and $\overline{V} \subset U$.

$C(U) = \{ f : U \to \mathbb{R}$ continuous $\}$

$C^m(U) = \{ f : U \to \mathbb{R} \quad D^\alpha f \in C(U) \quad \forall\, \alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in \mathbb{N}^n$
$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \le m \}$

$C^{0,1}(U) = \{ f \quad \exists\, C_f \quad |f(x) - f(y)| \le C_f |x-y| \quad \forall x, y \in U \}$ LIPSCHITZ FUNCTIONS

$C^{0,\alpha}(U) = \{ f \quad \exists\, C_f \quad |f(x) - f(y)| \le C_f |x-y|^\alpha \quad \forall x, y \in U \}$
$0 < \alpha < 1$ HÖLDER CONTINUOS

$C^{0,1}_{loc}(U) = \{ f \quad \forall K \subset\subset U$ compact $\exists\, C_{K,f}$
$|f(x) - f(y)| \le C_{K,f} |x-y| \quad x, y \in K \}$

def $f \in C^{0,1}(U)$ $\quad C_f$ = lipschitz constant of f := $\sup_{\substack{x,y\in U \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}$

$f \in C^{0,\alpha}(U)$ $\quad C$ = Hölder constant $=$

$$= \sup_{\substack{x\neq y \\ x,y\in U}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Obs $f \in C^{0,1}(U)$ $\quad \exists\, \bar{f} \in C^{0,1}(\mathbb{R}^n)$ LIPSCHITZ EXTENSION such that $\bar{f} = f$ on U of f

several lipschitz extension Let C_f the lipschitz constant of f

$$\bar{f}_1(x) = \min_{y\in U} \{f(y) + C_f|y-x|\}$$ is the smallest extension

$$\bar{f}_2(x) = \max_{y\in U} \{f(y) - C|y-x|\}$$

Also true for Hölder (by subadditivity of concave functions)

ASCOLI-ARZELÀ THEOREM (compactness theorem)

K compact $f_n \in \mathcal{C}(K)$ $(f_n)_n$ sequence

1) $\|f_n\|_\infty = \sup_{x\in K} |f_n(x)| \le C$
2) $\forall \varepsilon\ \exists \delta \quad |x-y| \le \delta \Rightarrow |f_n(x) - f_n(y)| \le \varepsilon$

] true $\forall n$ with the same $C>0$ and $\delta = \delta(\varepsilon)$.

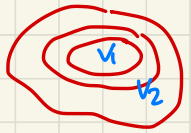
$\exists f_{n_k}$ such that $f_{n_k} \to f$ uniformly $f \in \mathcal{C}(K)$

that is $\|f_{n_k} - f\|_\infty = \sup_{x\in K} |f_{n_k}(x) - f(x)| \to 0$

EXTENSION

Let U be σ-compact that is $U = \bigcup_{k=1}^{\infty} V_k$ · $V_k \subset V_{k+1}$ · $V_k \subset\subset U$

(ex $V_k = \{x \in U \ \ d(x, \mathbb{R}^n \setminus U) > \frac{1}{k} \ \ |x| < k\}$)



If $f_n \in \mathcal{C}(U)$ · $\exists C>0$ $\sup_{x\in K} |f_n(x)| \le C$ $\forall n$

· $\forall \varepsilon\ \exists \delta = \delta(\varepsilon) \quad |x-y| \le \delta \Rightarrow |f_n(x) - f_n(y)| \le \varepsilon$ $\Longrightarrow$?

$\Rightarrow \exists\, f_{n_k}$ and $f \in \mathcal{C}(U)$ such that

$f_{n_k} \to f$ locally uniformly, that is

$\forall K \subset\subset U$ compact $\|f_{n_k} - f\|_\infty \to 0$.

(by A.A using a diagonalization argument)

$\sim\cdot\sim$

$(\mathcal{C}(K), \|\cdot\|_\infty)$ is a Banach space (by Ascoli-Arzelà)

NORM $\|f\|_\infty = \sup_{x \in K} |f(x)|$

$(\mathcal{C}(K), \|\cdot\|_\infty) \hookrightarrow (L^\infty(K), \|\cdot\|_\infty)$ measurable functions which are essentially bounded.

↳ closed subspace of

$(\mathcal{C}^m(K), \|\cdot\|_{\infty,m})$ is a Banach space (by Ascoli-Arzelà)

$\|f\|_{\infty,m} = \sup_{x \in K} \sup_{|\alpha| \le m} \|D^\alpha f\|_\infty$.

$$C^\infty(K) = \bigcap_{m \geq 0} C^m(K)$$

$$d(f,g) := \sum_{m=0}^{+\infty} \frac{2^{-m} \|f-g\|_m}{1+\|f-g\|_m}$$

$(C^\infty(K), d_K)$ is COMPLETE METRIC SPACE (FRÉCHET SPACE) (NO NORM! NOT A BANACH SPACE)

$f_n \to f$ in $C^\infty(K) \iff d_K(f_n, f) \to 0 \iff \|D^\alpha f_n - D^\alpha f\|_\infty \to 0 \quad \forall \alpha \in \mathbb{N}^n$

Obs: BOUNDED SETS ARE PRECOMPACT

$\Rightarrow$ (NEVER TRUE FOR INFINITE DIMENSIONAL BANACH SPACES!)

$\Rightarrow$ In a Banach space of INFINITE DIMENSION, CLOSED BALLS ARE NEVER COMPACT.

Let $f_n \in C^\infty(K)$ s.t. that $\exists C>0$ $d(f_n, 0) \leq C \Rightarrow$

$\exists f_{n_k}$ subsequence and $f \in C^\infty(K)$ such that

$d_k(f_{n_k}, f) \to 0$ (CONSEQUENCE of ASCOLI-ARZELÀ THM + DIAGONALIZATION ARGUMENT)