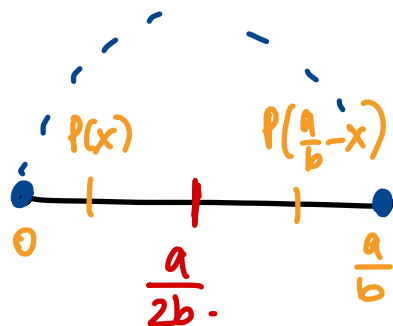


π è IRRAZIONALE (~ 1760)

Supponiamo per assurdo $\pi = \frac{a}{b}$, $a, b \in \mathbb{N}$

Useremo: $\sin(\pi) = \sin\left(\frac{a}{b}\right) = 0$

Considero $P(x) = \frac{1}{n!} x^n (a - bx)^n$



Fatti: P ha grado $2n$

P ha solo termini di grado $\geq n$.

P ha coefficienti in $\mathbb{Q}$.

In fine $P(x) = P\left(\frac{a}{b} - x\right) \quad \forall x \in [0, \frac{a}{b}]$.

$$\begin{aligned} P\left(\frac{a}{b} - x\right) &= \frac{1}{n!} \left(\frac{a}{b} - x\right)^n \left(a - b\left(\frac{a}{b} - x\right)\right)^n \\ &= \frac{1}{n!} \frac{(a - bx)^n}{b^n} (\cancel{a} - \cancel{a} + bx)^n \\ &= \frac{1}{n!} \frac{(a - bx)^n}{\cancel{b^n}} \cdot \cancel{b^n} \cdot x^n = P(x) \end{aligned}$$

$$P^{(i)}(0) = 0$$

$$i = 1, \dots, n-1.$$

$$P^{(j)}(0) \in \mathbb{Z} \quad = j! a_j \quad (**)$$

$$j = n, \dots, 2n$$

Per ogni $m \in \mathbb{N}$, dato $P(x) = P_m(x)$ continuo

$$Q(x) = P(x) - P''(x) + P^{(4)}(x) - P^{(6)}(x) + \dots + (-1)^m P^{(2m)}(x)$$

$$Q''(x) = P'' - P^{(4)} + P^{(6)} + \dots$$

• Hengro

$$Q''(x) = P(x) - Q(x)$$

molte ↓ grazie a (**)

$$Q(0) \in \mathbb{Z}$$

$$Q\left(\frac{a}{b}\right) \in \mathbb{Z}$$

↑ fu la simmetria.

Ora che

$$\begin{aligned} \frac{d}{dx} (Q'(x) \sin x - Q(x) \cos x) \\ &= Q''(x) \sin x + \cancel{Q'(x) \cos x} \\ &\quad - \cancel{Q'(x) \cos x} + Q(x) \sin x \\ &= (Q''(x) + Q(x)) \sin x = P(x) \sin x. \end{aligned}$$

$$\int_0^\pi P(x) \sin x \, dx = \left[Q'(x) \sin x - Q(x) \cos x \right]_{x=0}^{x=\pi}$$

(ma qui $\sin \pi = 0$) $= Q(\pi) - Q(0) \in \mathbb{Z}.$

Per ogni $n \in \mathbb{N}$

$$\int_0^\pi P(x) \sin x \, dx \cdot \text{ è intero } \underline{\text{diverso da zero}}$$

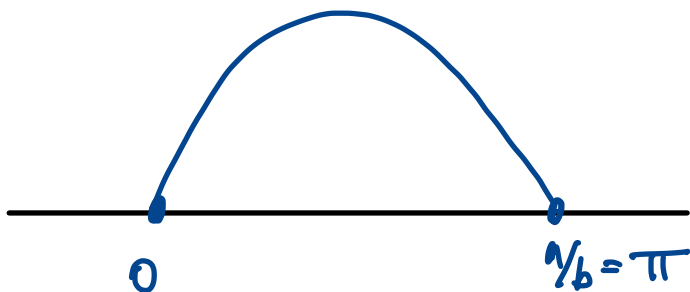
Orn vedremo che per n grande ciò è in contraddizione con la seg. proprietà:

$$P(x) = \frac{1}{n!} x^n (a-bx)^n \quad P(0) = 0 = P\left(\frac{a}{b}\right) \\ P(x) \geq 0.$$

$$\begin{aligned} P'(x) &= \frac{1}{n!} \left(n x^{n-1} (a-bx)^n + n x^n (a-bx)^{n-1} \cdot (-b) \right) \\ &= \frac{1}{(n-1)!} x^{n-1} (a-bx)^{n-1} \left[(a-bx) + (-bx) \right] \\ &= \frac{1}{(n-1)!} x^{n-1} (a-bx)^{n-1} [a - 2bx]. \end{aligned}$$

$$\text{Sull'intervallo }]0, \frac{a}{b}[\quad P'(x) = 0 \quad (\Rightarrow) \quad x = \frac{a}{2b}.$$

Dunque $P(x)$ ha massimo in $x = \frac{a}{2b}$.



Per $x \in [0, \frac{a}{b}]$

$$0 \leq P(x) \sin x \leq P\left(\frac{a}{2b}\right) \cdot 1$$

$$0 \leq P(x) \sin x \leq \frac{1}{n!} \left(\frac{a}{2b}\right)^n \left(a - b\left[\frac{a}{2b}\right]\right)^n$$

semplificando

$$0 \leq P(x) \sin x \leq \frac{\pi^n a^n}{4^n n!} \quad \forall x \in [0, \pi] \quad \text{con } \pi = \frac{a}{b}$$

$$0 \leq \int_0^{\pi} P(x) \sin x \, dx \leq \frac{\pi^n a^n}{4^n n!} \cdot \pi$$

$$\text{ma } \lim_{n \rightarrow +\infty} \frac{\pi^n a^n}{4^n n!} = 0$$

critério del
rapporto

Donque $\int_0^{\pi} P(x) \sin(x) \, dx$ non può essere
intero diverso da zero
per n grande.