



1. Utilizziamo i seguenti sviluppi di Taylor

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + o(y^4), \quad y \rightarrow 0$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4), \quad x \rightarrow 0$$

$$\cosh y = 1 + \frac{y^2}{2} + \frac{y^4}{24} + o(y^4), \quad y \rightarrow 0$$

$$\Rightarrow e^{1-\cos x} = 1 + (1-\cos x) + \frac{1}{2}(1-\cos x)^2 + o(x^4)$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{1}{2} \frac{x^4}{4} + o(x^4)$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{12} + o(x^4), \quad x \rightarrow 0$$

e anche

$$\cosh(\alpha x) = 1 + \alpha^2 \frac{x^2}{2} + \alpha^4 \frac{x^4}{24} + o(x^4), \quad x \rightarrow 0$$

$$\Rightarrow f(x) = e^{1-\cos x} - \cosh(\alpha x) = \frac{x^2}{2}(1-\alpha^2) + \frac{x^4}{12}\left(1-\frac{\alpha^4}{2}\right) + o(x^4)$$

in particolare per

$$\alpha = 1: \quad f(x) = \frac{x^4}{24} + o(x^4), \quad x \rightarrow 0$$

$$\alpha = 2: \quad f(x) = -\frac{3}{2}x^2 + o(x^2)$$

Precisamente ($\alpha > 0$)

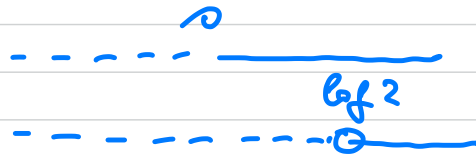
$$\text{se } \alpha = 1 \quad \lim_{x \rightarrow 0^+} \frac{x^\beta}{f(x)} = \begin{cases} 0, & \beta > 4 \\ 24, & \beta = 4 \\ +\infty, & \beta \in]0, 4[\end{cases}$$

$$\text{se } \alpha = 2 \quad \lim_{x \rightarrow 0^+} \frac{x^\beta}{f(x)} = \begin{cases} 0, & \beta > 2 \\ -\frac{2}{3}, & \beta = 2 \\ -\infty, & \beta \in]0, 2[\end{cases}.$$

2. $f(x) = -\arctan\left(\sqrt{\frac{e^x - 1}{e^x - 2}}\right)$

il dominio

$$\frac{e^x - 1}{e^x - 2} > 0, \quad x \neq \ln 2$$



$$\Rightarrow D =]-\infty, 0] \cup]\ln 2, +\infty[$$

inoltre: da $\lim_{x \rightarrow -\infty} \frac{e^x - 1}{e^x - 2} = \frac{1}{2}$

$$\Rightarrow \lim_{x \rightarrow -\infty} f(x) = -\arctan\left(\frac{1}{\sqrt{2}}\right)$$

$y = -\arctan\left(\frac{1}{\sqrt{2}}\right)$ è asintoto orizzontale
per $x \rightarrow -\infty$

$$\lim_{x \rightarrow +\infty} \frac{e^x - 1}{e^x - 2} = 1 \Rightarrow \lim_{x \rightarrow +\infty} f(x) = -\arctan(1) = -\frac{\pi}{4}$$

$y = -\frac{\pi}{4}$ è asintoto orizzontale per $x \rightarrow +\infty$;

$$\lim_{x \rightarrow \log 2^+} \frac{e^x - 1}{e^x - 2} = +\infty \Rightarrow \lim_{x \rightarrow \log 2^+} f(x) = -\frac{\pi}{2}$$

Le derivate:

$$f'(x) = - \frac{1}{1 + \left(\frac{e^x - 1}{e^x - 2}\right)} \cdot \frac{1}{2} \sqrt{\frac{e^x - 2}{e^x - 1}} \cdot \frac{\overbrace{e^x(e^x - 2) - (e^x - 1)e^x}^{= -e^x}}{(e^x - 2)^2}$$

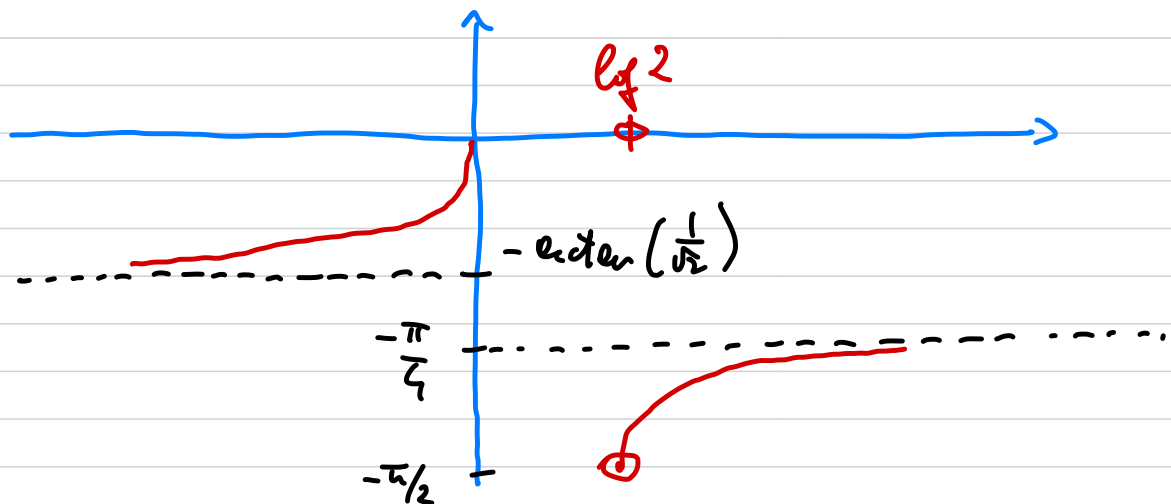
$$\Rightarrow f'(x) > 0 \quad \forall x \in \mathbb{D} \setminus \{0\}.$$

f è strettamente crescente in $] -\infty, 0]$ e in $] \log 2, +\infty [$.

Calcoliamo inoltre

$$\lim_{x \rightarrow 0^-} f'(x) = +\infty \quad \text{e} \quad \lim_{x \rightarrow \log 2^+} f'(x) = +\infty.$$

Il grafico risulta dunque:



3. Risolvere $z^3 = (\bar{z})^2$

prendiamo il modulo

$$|z|^3 = |\bar{z}|^2 = |z|^2 \Rightarrow |z|^2(|z| - 1) = 0$$

$\Rightarrow$

- $|z| = 0 \quad \Rightarrow \quad z = 0$
- oppure

- $|z| = 1$

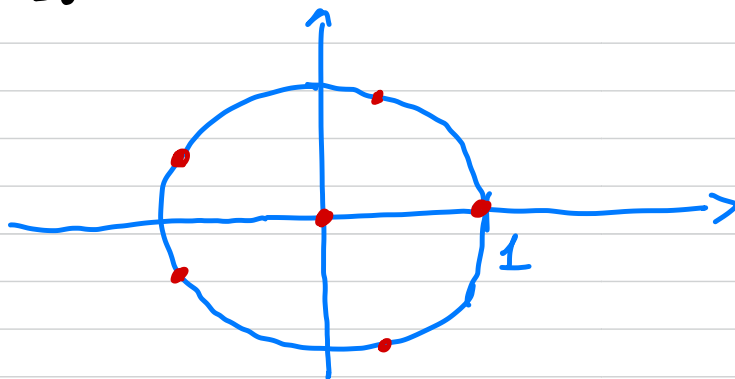
nel secondo caso moltiplichiamo per z^2 :

$$z^5 = (\bar{z})^2 \cdot z^2 = |z|^4 = 1$$

$$\Rightarrow z^5 = 1 = \cos 0 + i \sin 0 \quad \text{radici quinte di 1.}$$

$$\Rightarrow \operatorname{Arg} z = \frac{2\pi}{5} k, \quad k = 0, 1, \dots, 4$$

Nel piano complesso



4. $\int \frac{x^3 + x + 1}{x^2 + 1} \arctan x \, dx =$ (divisione)

$$= \int x \arctan x + \frac{1}{1+x^2} \arctan x \, dx =$$

$$= \underbrace{\frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2 \pm 1}{1+x^2} dx}_{\text{per parti}} + \frac{1}{2} \operatorname{arctg}^2 x =$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{x}{2} + \frac{1}{2} \operatorname{arctg} x + \frac{1}{2} \operatorname{arctg}^2 x + c, c \in \mathbb{R}$$

$$= F_c(x).$$

Imponiamo allora

$$2 = F_c(0) = c$$

perciò la primitiva cercata è:

$$F_2(x) = \frac{1}{2} \left((x^2+1) \operatorname{arctg} x - x + \operatorname{arctg}^2 x \right).$$

Per il teorema fondamentale del calcolo

$$\int_0^1 \frac{x^3+x+1}{x^2+1} \operatorname{arctg} x \, dx = F_2(1) - F_2(0) =$$

$$= \operatorname{arctg} 1 - \frac{1}{2} + \frac{1}{2} \operatorname{arctg}^2 1 = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi^2}{32}.$$