Problem sheet 3: Distribution theory
Functions theory

Exercise 1.

(1) Let $f \in L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} f(x)dx = \alpha$. Show that $f_n(x) = n^N f(xn) \to \alpha\delta_0$ in $\mathcal{D}'(\mathbb{R}^N)$.
(2) Let $f_n \in L^p(\mathbb{R}^n)$, $p > 1$, with $\|f_n\|_p \leq C$ and such that there exists f measurable s.t. $f_n \to f$ in the sense of distribution (that is $T_{f_n} \to T_f$). Is it true that $f \in L^p(\mathbb{R}^n)$ and $f_n \to f$ weakly in L^p? (recall the weak compactness theorem..).
(3) Is the previous statement true for $p = 1$?

Exercise 2. Let $T \in \mathcal{D}'(I)$ for some interval $I \subseteq \mathbb{R}$. Show that $T' = 0$ in the sense of distribution (this means $T'\phi = -\int T\phi' dx = 0$) if and only if there exists a constant c and $T = c$ (this means $T\phi = c\int \phi dx$).
Hint: argue as in the fundamental lemma of the calculus of variations.

Exercise 3.
Let $T \in \mathcal{D}'(U)$ be a positive distribution, that is $T(\phi) \geq 0$ for every test $\phi \geq 0$. Prove that T has order 0.
Hint: recall the property of positive linear functionals on $C_c(U)$

Exercise 4. Let $n > 1$ and $E \subseteq \mathbb{R}^n$ be a bounded open set with boundary of class C^1 and consider the distribution

$$T_{\chi_E}(\phi) := \int_E \phi(x)dx.$$

Compute the distributional gradient of T_{χ_E}.