**Def: [Infinite divisible law (IDL)]**
A random variable $X$ with value in $\mathbb{R}^d$ has IDL if $\forall k \geq 1$, there exist $k$ iid r.v.'s $X_1^{(k)}, \dots, X_k^{(k)}$ st. $X \overset{d}{=} X_1^{(k)} + \dots + X_k^{(k)}$ (*)

<u>Note</u>: In terms of characteristic functions, (*) is equivalent to

$$\varphi_X(\theta) = \varphi_{X_1^{(k)} + \dots + X_k^{(k)}}(\theta) = \left(\varphi_{X_1^{(k)}}(\theta)\right)^k, \quad \text{for } \theta \in \mathbb{R}^d$$

(where $\varphi_Y(\theta) = \mathbb{E}(e^{iY\cdot\theta})$ characteristic function of $Y$).

Examples:

1. $X = b \in \mathbb{R}^d$ a.s. $\Rightarrow \varphi_X(\theta) = e^{ib\theta} = \left(e^{i\frac{b}{k}\theta}\right)^k$

   and (*) holds with $X_1^{(k)} = \frac{b}{k}$ a.s.

2. $X \sim N(0, \sigma^2)$ in $\mathbb{R} \Rightarrow \varphi_X(\theta) = e^{-\frac{1}{2}t^2\sigma^2} = \left(e^{-\frac{1}{2}t^2\frac{\sigma^2}{k}}\right)^k$

   and (*) holds with $X_1^{(k)} \sim N\left(0, \frac{\sigma^2}{k}\right)$ (and similarly in $\mathbb{R}^d$)

3. $X \sim Poi(\lambda) \Rightarrow \varphi_X(\theta) = e^{(\lambda(e^{it}-1))} = \left(e^{(\frac{\lambda}{k}(e^{it}-1))}\right)^k$

   and (*) holds with $X_1^{(k)} \sim Poi\left(\frac{\lambda}{k}\right)$

4. Let $Z = \sum_{j=1}^{N} X_j$, $(X_j)_{j\in\mathbb{N}}$ iid on $\mathbb{R}^d$ with law $\nu$, $N \sim Poi(\lambda)$ indep. of the $X_j$'s. Then $Z$ has compound Poisson law $CP(\lambda, \nu)$

$$\varphi_Z(t) = \mathbb{E}\left(\varphi_{X_1}(t)^N\right) = \sum_{m=0}^{\infty} \varphi_{X_1}(t)^m \cdot \frac{\lambda^m}{m!} e^{-\lambda t} = e^{(\lambda(\varphi_{X_1}(t)-1))}$$

$\nu$ is the law of $X_1$

$$= e^{\lambda\left(\int_{\mathbb{R}^d}(e^{ity}-1)\nu(dy)\right)} = \left(e^{\frac{\lambda}{k}\left(\int_{\mathbb{R}^d}(e^{ity}-1)\nu(dy)\right)}\right)^k$$

   and (*) holds with $Z^{(k)} \sim CP(\lambda/k, \nu)$

5. $X \sim Cauchy(c)$ on $\mathbb{R}$ so that the density $f(x) = \frac{c}{\pi} \cdot \frac{1}{x^2+c^2}$

$$\Rightarrow \varphi_X(\theta) = e^{c|\theta|} = \left(e^{\frac{c}{k}|\theta|}\right)^k$$

   and (*) holds with $X^{(k)} \sim Cauchy\left(\frac{c}{k}\right)$

Going back to Levy processes, it holds

**Lemma**: If $(X_t)_{t\geq 0}$ is a LP, then $X_t$ has IDL, $\forall t\geq 0$.

Moreover: $(\bullet)$ $\boxed{\varphi_{X_t}(\theta) = e^{t\psi(\theta)}}$

where $\psi(\theta) = \log \varphi_{X_1}(\theta)$ is called characteristic exponent of the LP.

**Proof**: $\forall n\geq 1$, divide $[0,t]$ in $n$ intervals of length $t/n$, so that

$$X_t = \sum_{k=1}^{n} \underbrace{\left(X_{k\frac{t}{n}} - X_{(k-1)\frac{t}{n}}\right)}_{\text{are iid} \overset{d}{=} X_{t/n}} \qquad \text{so that} \quad \varphi_{X_t}(\theta) = \left(\varphi_{X_{t/n}}(\theta)\right)^{n} \ (*) \quad \overset{(IDL)}{\checkmark}$$

Taking $m\geq 1$ and using $(*)$ twice: $\left(\varphi_{X_1}(\theta)\right)^{m} = \varphi_{X_m}(\theta) = \left(\varphi_{X_{m/n}}(\theta)\right)^{n}$

$\Longrightarrow \varphi_{X_{m/n}}(\theta) = \left(\varphi_{X_1}(\theta)\right)^{m/n} \Longrightarrow X_t$ satisfies $(\bullet)$ for $t\in\mathbb{Q}^+$.

For $t\in\mathbb{R}^+\setminus\mathbb{Q}^+$, consider $(t_n)_{n\in\mathbb{N}}\in\mathbb{Q}^+$ s.t. $t_n\downarrow t$ as $n\to\infty$.

Since $(X_t)_{t\geq 0}$ is right-cont. a.s., also $\varphi_{X_t}(\theta)$ is right-cont (in $t$) by dominated convergence, and then

$$\varphi_{X_t}(\theta) = \lim_{t_n\downarrow t} \varphi_{X_{t_n}}(\theta) = \lim_{t_n\downarrow t}\left(\varphi_{X_1}(\theta)\right)^{t_n} = \left(\varphi_{X_1}(\theta)\right)^{t} = e^{t\psi(\theta)} \quad (\bullet)\checkmark$$

#

- Law of a LP:

The finite dimensional distributions of the coordinates $(X_{t_1},\dots,X_{t_m})$, are given has a function of the joint-distributions of the increments $(X_{t_1}, X_{t_2}-X_{t_1},\dots,X_{t_m}-X_{t_{m-1}})$.

Since LP have stationary and independent increments, this is simply a product measure $\bigtimes_{k=1}^{m} \pi_{t_k-t_{k-1}}$, with $\pi_t \sim X_t$.

At last, since the law $\pi_t$ is identified by the characteristic function $\varphi_{X_t}(\theta) = e^{t\psi(\theta)}$, the law of the whole process is identified by the characteristic exponent $\psi$

with $\psi(\theta) = \log\left(\varphi_{X_1}(\theta)\right)$, $\theta\in\mathbb{R}^d$

with $\Psi(\theta) = \log(\varphi_{X_1}(\theta))$ , $\theta \in \mathbb{R}^d$

where $X_1$ has IDL (as well as all the coordinates $X_t$, $t>0$).

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The main point, is that the characteristic exponent of a IDL takes an explicit expression depending on three parameters. This general expressions hence characterize the law of any general LP.

**Theorem** [Lévy-Khintchine formula] (without proof)

A random variable $X$ has IDL with characteristic exponent $\Psi$, namely $\varphi_X(\theta) = e^{\Psi(\theta)}$ with $\theta \in \mathbb{R}^d$, if and only if $\exists$ a triple $(b, \sigma^2, \Lambda)$ where $b \in \mathbb{R}^d$, $\sigma^2$ is a symmetric, positive definite $M_{d,d}$ , and $\Lambda$ is a measure on $\mathbb{R}^d \setminus \{0\}$ with $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \Lambda(dx) < \infty$, such that

$$\Psi(\theta) = i b \cdot \theta - \frac{1}{2} \theta \cdot \sigma^2 \theta + \int_{\mathbb{R}^d} \left(e^{i\theta \cdot x} - 1 - i\theta \cdot x \mathbb{1}_{B_1}(x)\right) \Lambda(dx), \quad \forall \theta \in \mathbb{R}^d$$

Note:
- $(b, 0, 0) \longrightarrow X = b$
- $(b, \sigma^2, 0) \longrightarrow X \sim N(b, \sigma) = b + N(0, \sigma)$
- $(0, 0, \lambda \cdot \delta_1) \longrightarrow X \sim Poi(\lambda)$ (in $\mathbb{R}^+$)
- $(0, 0, \lambda \cdot \nu) \longrightarrow X \sim CPoi(\lambda, \nu)$, where $\nu \in \mathcal{P}(\mathbb{R}^d \setminus B_1)$

Equiv. $\tilde{\nu} = \lambda \nu$ with $\tilde{\nu} \in \mathcal{M}(\mathbb{R}^d)$ finite s.t. $\tilde{\nu}(B_1) = 0$,

where $\lambda = \tilde{\nu}(\mathbb{R}^d)$ and $\nu = \frac{\tilde{\nu}}{\lambda} \in \mathcal{P}(\mathbb{R}^d \setminus B_1)$

As an application of the Lévy-Khintchine formula, one gets the following characterization of LP.

**Theorem** [Lévy-Ito decomposition]

For any triple $(b, \sigma^2, \Lambda)$, $b \in \mathbb{R}^d$, $\sigma^2 \in M_{d \times d}$ symmetric, definite positive and $\Lambda$ on $\mathbb{R}^d \setminus \{0\}$ s.t. $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \Lambda(dx) < \infty$, there exists a probability space on which are defined $X^{(1)}, X^{(2)}, X^{(3)}$ which are independent Lévy process such that:

probability space on which are defined $X^{(1)}, X^{(2)}, X^{(3)}$ which are <u>independent</u> Lévy processes such that:

- $X^{(1)}_t = bt + \sigma B_t$ Brownian motion with drift (continuous)

  $\hookrightarrow$ with $\psi^{(1)}(\theta) = ib\theta - \frac{1}{2}\theta \cdot \sigma^2 \theta$

- $X^{(2)}_t = \sum_{k=1}^{N_t} Y_k$ Compound Poisson process $\left\{ \begin{array}{l} (N_t) \sim PP(\Lambda(B^c)) \\ (Y_k)_{k \in \mathbb{N}} \text{ iid s.t} \\ Y_k \sim \frac{\Lambda(dx)}{\Lambda(B^c)} \mathbb{1}_{\{x \in B^c\}} \end{array} \right.$

  $\hookrightarrow$ with $\psi^{(3)}(\theta) = \int_{B^c} (e^{i\theta x} - 1) \Lambda(dx)$

- $X^{(3)}_t$ = square integrable martingale with at most contable jumps on any finite interval of time.

  $\hookrightarrow$ with $\psi^{(2)}(\theta) = \int (e^{i\theta x} - 1 - i\theta x) \Lambda(dx)$

In particular $X = X^{(1)} + X^{(2)} + X^{(3)}$ is a Lévy process with characteristic exponent $\psi(\theta)$ as in $(*)$.

<u>Interpretation</u>: $(X^{(3)}_t)_{t \geq 0}$ is obtained by superposing independent CP with drift $\longrightarrow$ it can be shown it is a <u>square-integrable martingale</u>.

<u>Idea</u>: Let $D_m = \{ x : 2^{-(m+1)} \leq |x| < 2^{-m} \}$, $\lambda_m = \Lambda(D_m)$ and $\nu_m(dx) = \frac{\Lambda(dx)}{\lambda_m} 1_{D_m}$

so that $\int_{B_1} (e^{i\theta x} - 1 - i\theta x) \Lambda(dx) = \sum_{m \geq 0} \left( \lambda_m \int_{D_m} (e^{i\theta x} - 1 - \underbrace{i\theta x}_{\text{drift corrector}}) \nu_m(dx) \right)$

Each term of the sum corresponds to the characteristic exponent of a $CPP(\lambda_m, \nu_m)$ with a drift term (or corrector), which turns the process into a (square integrable) martingale.

## <u>Comment</u>

A measure $\Lambda$ on $\mathbb{R}^d \setminus \{0\}$ s.t. $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \Lambda(dx) < \infty$ is called <u>Lévy measure</u>. Notice that the condition on $\Lambda$ corresponds to

$$(c) \left\{ \begin{array}{l} \bullet \ \Lambda(B_1^c) < \infty, \text{ where } B_1^c = \{ x \in \mathbb{R}^d : |x| \geq 1 \} \\ \bullet \ \int_{B_1} |x|^2 \Lambda(dx) < \infty \end{array} \right.$$

If [illegible] of $\Lambda$ is finite, then (c) is satisfied.

$B_1^\smile$

In particular, if $\Lambda$ is finite, then (c) is satisfied.
But one can consider the case in which $\Lambda$ is an infinite measure concentrated in $B_1$ (around 0), but still square-integrable.

## Lévy-Ito decomposition: representation of compound Poisson processes

Let $\Lambda$ be a Lévy measure on $\mathbb{R}^d \setminus \{0\}$.

$\boxed{1.}$ Compound Poisson Process on $B_1^c \subset \mathbb{R}^m$:

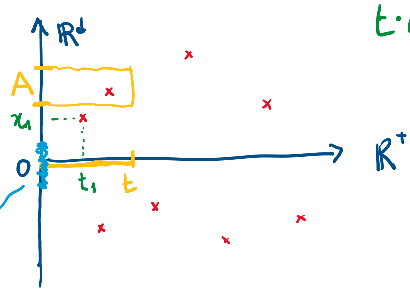
let $\lambda = \Lambda(B_1^c)$, and $\nu = \frac{\Lambda|_{B_1^c}}{\lambda} \longrightarrow \nu \in \mathcal{P}(B_1^c)$

a. Consider a Poisson random measure $N$ on $(\mathbb{R}^+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\mathbb{R}^d))$ with intensity $ds \times \lambda \cdot \nu(dx)$ (or equiv. $ds \times \Lambda|_{B_1^c}(dx)$)

Recall that this implies that, given $[0,t] \times A \in \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(B_1^c)$,

$$\mathbb{E}\left(N([0,t] \times A)\right) = \int_0^t ds \int_A \lambda\nu(dx) = \underbrace{t \cdot \lambda\nu(A)}_{= \, t \cdot \Lambda(A)} = \text{number of points in } A \text{ during time } [0,t]$$

Graphically




No arrivals on $B_1$

b. Each point $(t_1, x_1)$ of $N$ we associate a jump at time $t_1$ and of size $x_1$ of the corresponding $CPP(\lambda, \nu)$. Then we get

$$X_t := \int_{[0,t]} \int_{B_1^c} x \cdot N(ds, dx) \quad , \quad t \geq 0$$

This is indeed an alternative representation of CPP, as can be verified by computing $\varphi_{X_t}(\theta) = \mathbb{E}\left(e^{i\theta \cdot X_t}\right)$, as done below.

Note that, by definition, $X_t = N(x \cdot 1\!\!1_{[0,t]})$ $\quad (N(f))$

The computation of $\varphi_{X_t}(\theta)$ is then analogous to the computation of the Laplace functional $\mathcal{L}_f(N) = \mathbb{E}\left(e^{-N(f)}\right)$, and we get of the Laplace functional $\mathcal{L}_f(N) = \mathbb{E}(e^{\cdot\,\cdot\,\cdot})$, and we get

$$\varphi_{X_t}(\theta) = \exp\left(\int_0^t \int_{B_1^c} (e^{i\theta x} - 1)\, ds \cdot \lambda\nu(dx)\right)$$

$$= \exp\left(t\cdot\lambda \int_{B_1^c} (e^{i\theta x} - 1)\,\nu(dx)\right)$$

$$\Rightarrow \psi(\theta) = \lambda \int_{\mathbb{R}^d} (e^{i\theta\cdot x} - 1)\,\nu(dx), \quad \text{as wanted}$$

$\boxed{2.}$ Compound Poisson Process with drift on $D \equiv D_m \subset B_1$

Let $X_t := \int_0^t \int_D x\, N(ds, dx) - \underbrace{t\int_D x\,\Lambda(dx)}_{\text{drift}}$

↳ Note that this may be infinite, as well as $\Lambda(B_1)$

where $N$ is a PPP$(ds \times \Lambda(dx))$. In particular:

* $\mathbb{E}(X_t) = \mathbb{E}\left(N\left(x\cdot \mathbb{1}_{[0,t]\times D}\right)\right) - t\int_D x\,\Lambda(dx)$

* $\mathbb{E}(N(f)) = \mathrm{Leb}\times\Lambda(f)$

Properties of the PPP

$$\overset{*}{=} \int_{[0,t]}\int_D x\, ds\cdot\Lambda(dx) - t\int_D x\,\Lambda(dx) = 0$$

* $\mathbb{E}(X_t - X_s \mid \mathcal{F}_s) = \mathbb{E}(X_{t-s})$

since PPP has independence on disjoint sets and it is homogeneous on time

$$= \mathbb{E}\left(\int_{[s,t]}\int_D x\,N(ds,dx) - (t-s)\int_D x\cdot\Lambda(dx)\right) \overset{\text{same computation as before}}{=} 0$$

$\Rightarrow (X_t)_{t\geq 0}$ is a martingale

* Since $\int_D (|x|^2 \wedge 1)\,\Lambda(dx) < \infty$, $(X_t)$ is a square integrable martingale.

Indeed, using that for $f \in L^2(\Lambda) \cap L^1(\Lambda)$:

$\mathrm{Var}(N(f)) = \int f^2(s,x)\, ds\,\Lambda(dx)$ and that $\mathbb{E}(X_t) = 0$

we get $\mathbb{E}(X_t^2) = t\int_D x^2\,\Lambda(dx) < \infty$

**Remark**: If $\Lambda$ satisfies the stronger condition $\int_B |x|\,\Lambda(dx) = a < \infty$ then the drift term is precisely "$a\cdot t$", namely a proper drift.

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Short resume:

- If $X$ has IDL, by the Lévy-Khintchine formula,

$$\psi(\theta) = \left[\lg \varphi_X(\theta)\right] = \underbrace{ib\cdot\theta - \tfrac{1}{2}\theta\cdot\sigma^2\theta}_{} + \int_{\mathbb{R}^d} \underbrace{\left(e^{i\theta x} - 1 - i\theta x\,\mathbb{1}_{B_1}(x)\right)\Lambda(dx)}_{}$$

$$\psi(\theta) = \left[\lg \varphi_X(\theta)\right] = \underbrace{i b\cdot\theta - \tfrac{1}{2}\theta\cdot\sigma^2\theta}_{a} + \int_{\mathbb{R}^d} \underbrace{\left(e^{i\theta x} - 1 - i\theta x \mathbb{1}_{B_1}(x)\right)\Lambda(dx)}_{b}$$

with $(b, \sigma^2, \Lambda)$ canonical triple, $\Lambda$ s.t. $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\Lambda(dx) < \infty$

- A Lévy process $(X_t)_{t\geq 0}$ is s.t. $X_t$ has IDL with $\varphi_{X_t}(\theta) = e^{t\psi(\theta)}$
- The term a. corresponds to the charact. exp. of a BM with drift.

$$\longrightarrow \quad X_t^{(1)} = bt + \sigma^2 \cdot B_t$$

- The term b. can be splitted in two parts:

$$* \quad \int_{B_1^c} (e^{i\theta x} - 1)\Lambda(dx) = \lambda \int_{\mathbb{R}^d} (e^{i\theta x} - 1)\nu(dx) = \text{characteristic exponent of } CPP(\lambda, \nu)$$

since $\Lambda(B_1^c) < \infty$, set $\lambda = \Lambda(B_1^c)$ and $\nu = \frac{\Lambda_{|B_1^c}}{\lambda} \in \mathcal{P}(B_1^c)$

$$\longrightarrow \quad X_t^{(2)} = \sum_{k=1}^{M_t} Y_k \quad , \text{ with } (M_t) \sim PP(\lambda) \perp (Y_k) \text{ iid} \sim \nu$$

$$= \int_0^t \int_{B_1^c} x \cdot N(ds, dx) \quad , \text{ with } N \text{ a } PPP(\text{leb} \times \lambda\nu)$$

$$* \quad \int_{B_1} (e^{i\theta x} - 1 - i\theta x)\Lambda(dx)$$

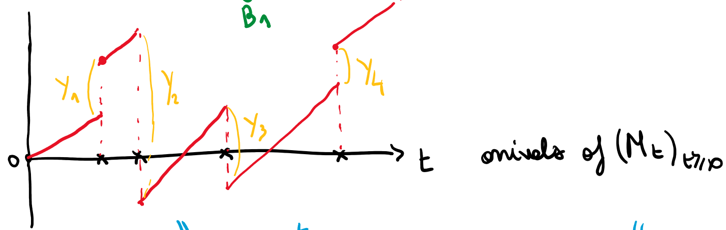
- if $\Lambda(B_1) < \infty \longrightarrow$ = characteristic exponent of CPP with drift:

$$X_t^{(3)} = \sum_{k=1}^{M_t} Y_k - t\int_{B_1} x\,\Lambda(dx)$$

$$= \int_0^t \int_{B_1} x\, N(ds, dx) - t\int_{B_1} x\,\Lambda(dx)$$

$(M_t), (Y_k), N$ as before, but defined on $B_1$

graphically




- if $\Lambda(B_1) = \infty \longrightarrow X_t$ " $= \lim_{\varepsilon \downarrow 0} \int_0^t \int_{D_\varepsilon} x\, N(ds, dx) - t\int_{D_\varepsilon} x\,\Lambda(dx)$ "

where $D_\varepsilon = \{x \in \mathbb{R}^d : \varepsilon < |x| < 1\}$ so that $\Lambda(D_\varepsilon) < \infty$

and the limit is taken uniformly over any interval $[0,T]$, with probability 1.

In conclusion, a general LP $(X_t)_{t\geq 0}$ is given by

In conclusion, a general LP $(X_t)_{t\geq 0}$ is given by

$$X_t = \overbrace{bt + \sigma B_t}^{\text{continuous process}} + \overbrace{\int_0^t\int_{B_1^c} x\cdot N(ds,dx) + \int_0^t\int_{B_1} x\cdot N(ds,dx) - t\int_{B_1} x\,\Lambda(dx)}^{\text{jump process - right continuous}}$$

bounded variation process

martingale

**Remark**: The sum of a (local) martingale and a bounded variation process is called semimartingale, and from the above construction, it turns out that LP's are semimartingales. The theory of stochastic integration (that you have seen for Ito processes - which are semimartingale) allows to extend the Ito calculus to semimartingale. In this sense, LP's are good integrators $dX_t$.