Computational imaging & learning: from physics to data-driven methods

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Mini-Corso Data Science @ UniPD

Università degli studi di Padova

May 11-14 2026

Practical information

My academic ID & Contacts

Luca Calatroni:
PI of the **Computational Imaging & Learning** team, MaLGa, UniGe.

- **MSc in Applied Mathematics**, 2011.
- **PhD in Applied Mathematics**, University of Cambridge, 2015.
- **Marie-Curie Post-doc** University of Genova, 2015-2016.
- **Post-doctoral fellowship**, *Lecteur Hadamard*: CMAP, École Polytechnique, France, 2016-2019.
- **CNRS researcher**, I3S lab, Sophia-Antipolis, France, 2019-2024.
- **Professore Associato**, Computer Science department, Genoa, Italy.

Contacts

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- **web**: `https://sites.google.com/view/lucacalatroni/home`

Course objectives and organisation

Objectives: understanding basic notions on forward/inverse modelling of exemplar computational imaging applications in biomedical imaging.

Approach: theory (8h) & Python labs (4h), on your laptop using DeepInverse library.

Syllabus

- **May 11, 14:30-16:30**, Theory: Introduction, forward/inverse imaging problems, regularisation.
- **May 12, 8:30-10:30**, Theory: Regularisation concepts, some practical computational imaging problems.
- **May 12, 10:30-12:30**, Lab: Regularisation for CT imaging.
- **May 13, 14:30-16:30**, Theory: Optimisation tools for computational imaging.
- **May 13, 16:30-18:30**, Lab: Optimisation-driven learned regularisation for CT imaging.
- **May 14, 8:30-10:30**, Theory: Advanced optimisation-driven learning for image reconstruction.

Main references

Books

C. A. Bouman, *Foundations of Computational Imaging: A Model-Based Approach*, SIAM, 2022.

G. Aubert, P. Kornprobst, *Mathematical Problems in Image Processing: PDEs and the COV*, II edition, Springer, 2000.

Tony F. Chan, J. Shen, *Image Processing and Analysis: variational, PDE, wavelet and stochastic methods*, SIAM, 2005.

Blogs

Tristan van Leeuwen: https://tristanvanleeuwen.github.io/IP_and_Im_Lectures/what_is.html

Johannes Meyer: https://computational-imaging.de/book/intro.html

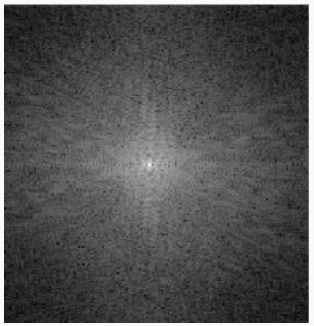
Computational resources

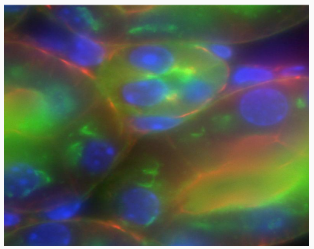
- Numerical tours by G. Peyré: http://www.numerical-tours.com/
- DeepInverse tutorials: https://deepinv.github.io/deepinv/auto_examples/index.html

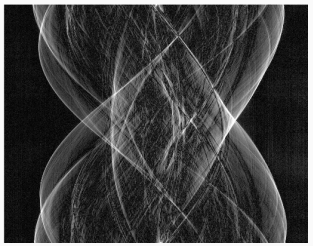
What is computational imaging ?

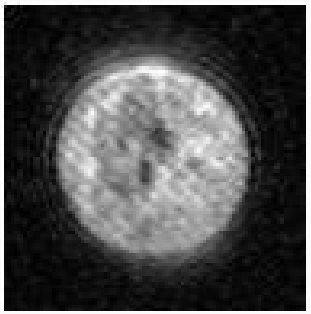
Microscopy

Medical Imaging









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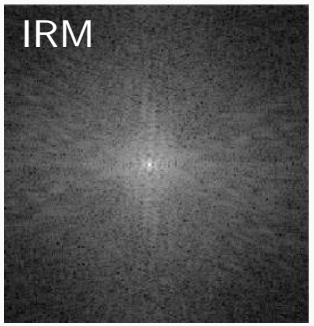

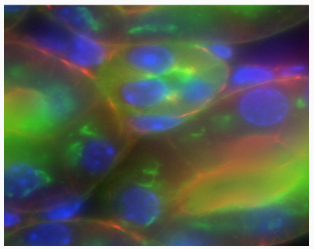

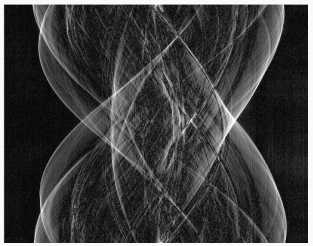

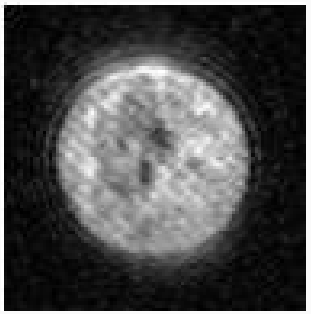

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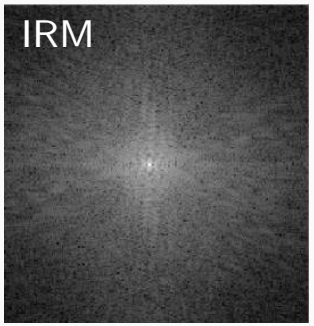

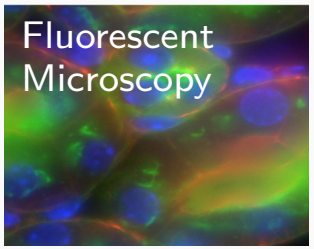

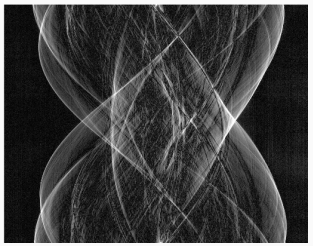

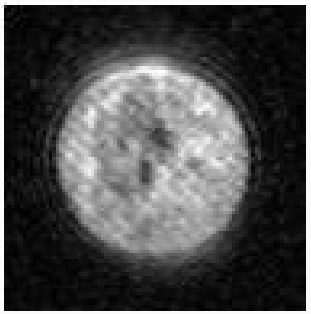

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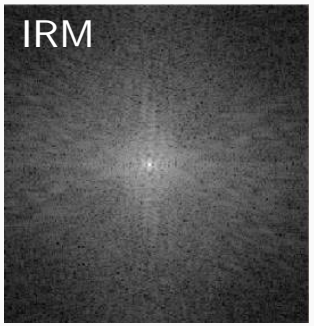

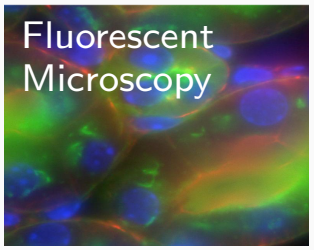

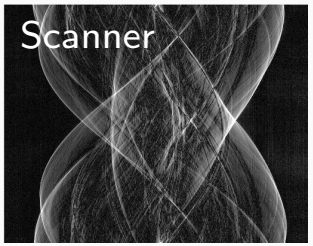

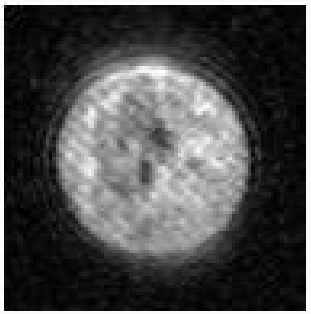

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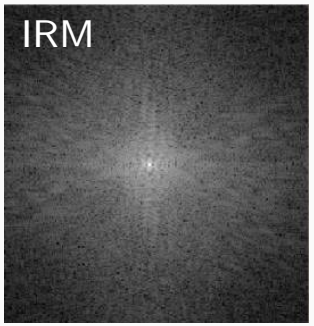

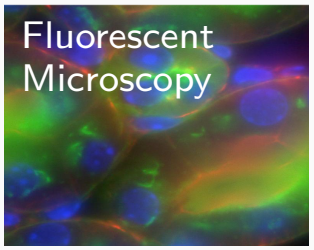

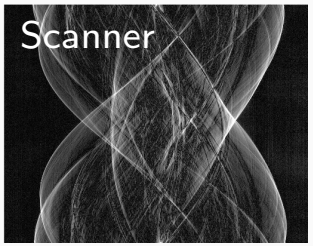

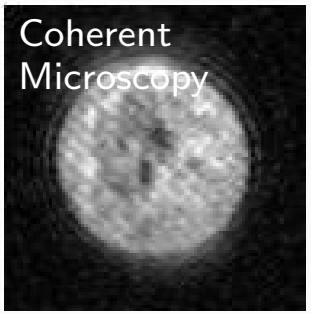

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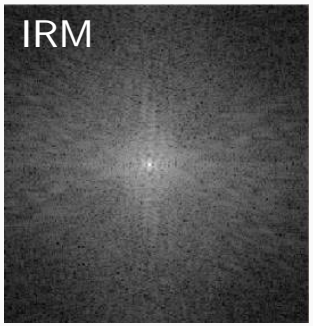

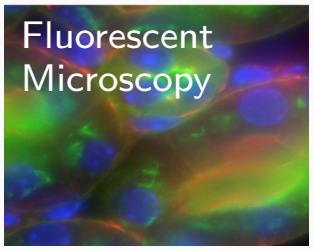

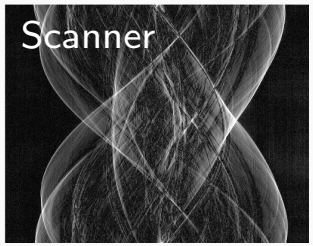

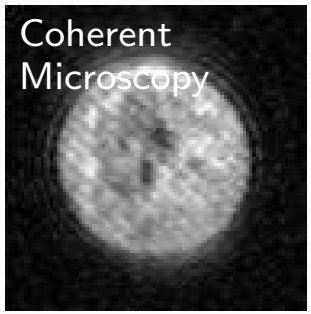

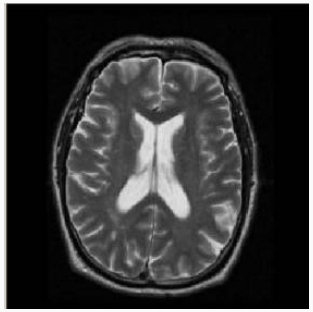

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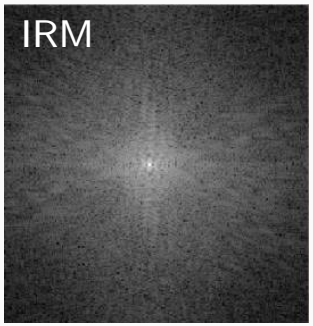

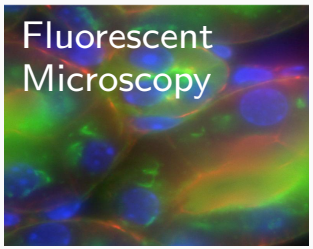

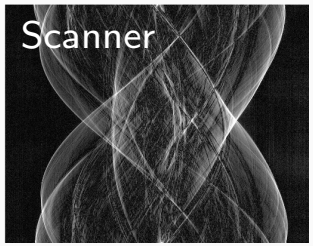

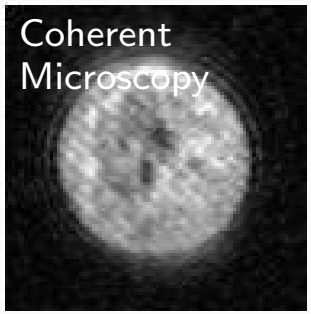

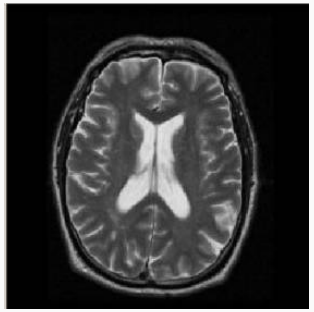

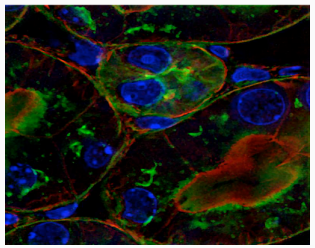

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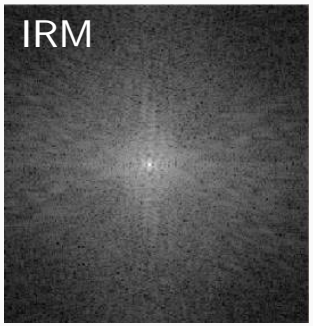

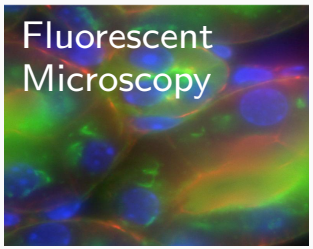

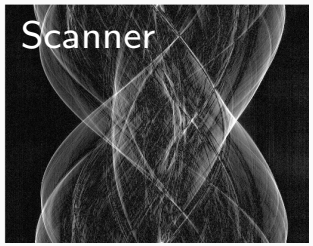

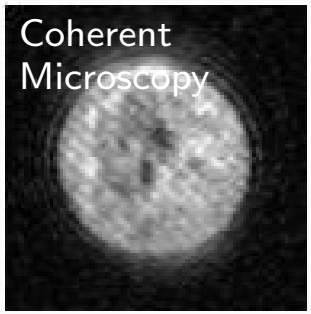

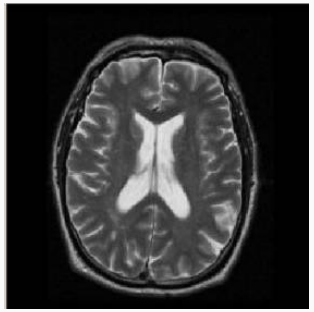

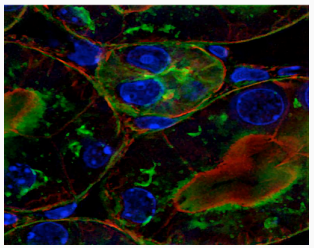

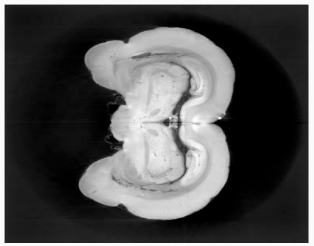

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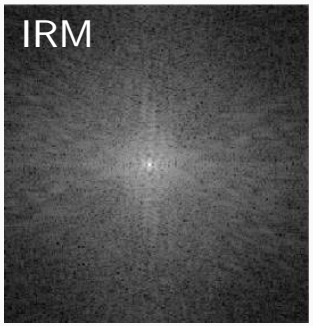

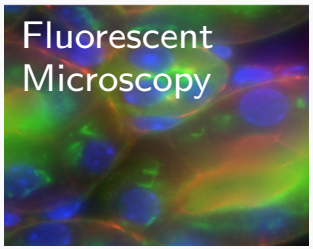

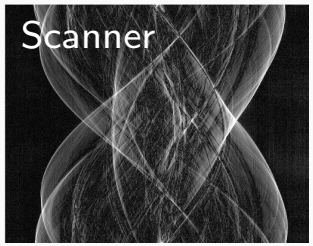

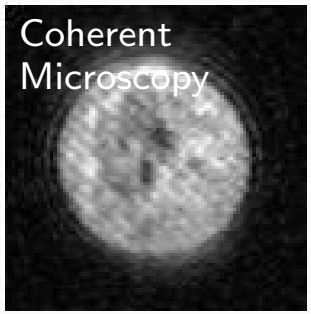

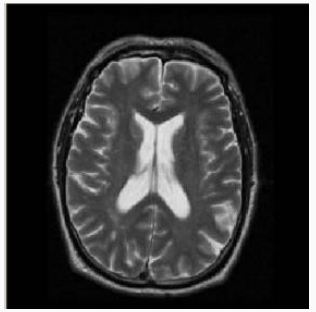

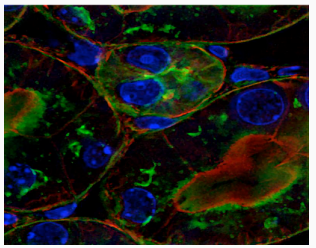

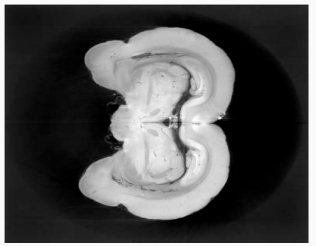

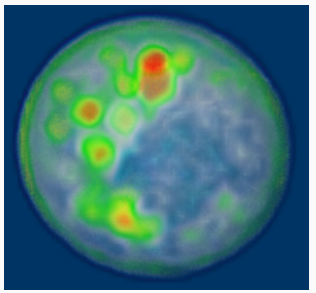

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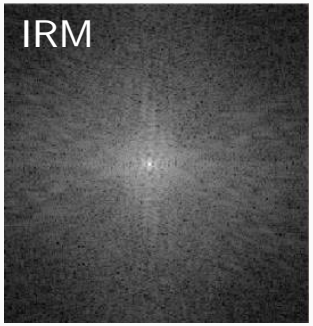

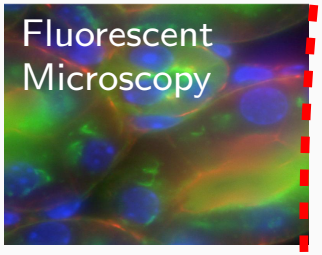

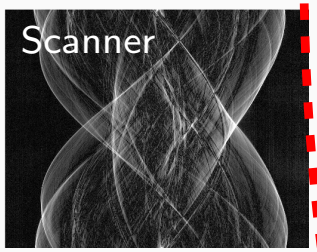

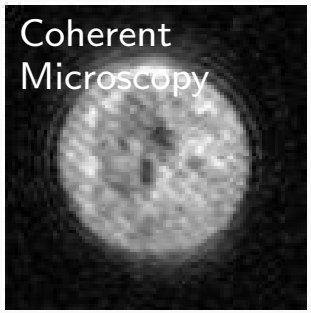

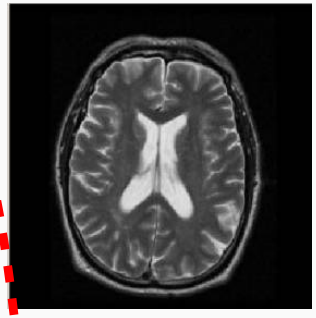

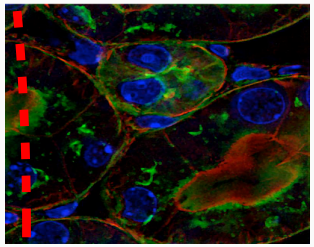

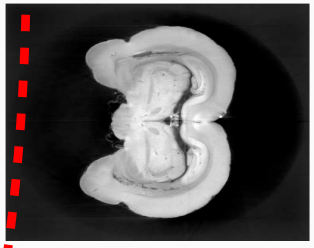

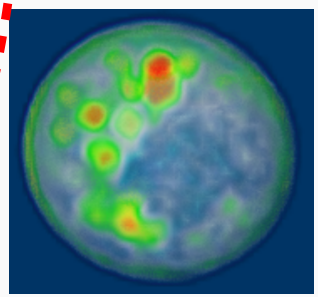

Outline

Practical information

From continuous to discretised image formation models

Ill-posedness

Maximum likelihood, MAP and MMSE

Regularisation: from priors to discrete models

From continuous to discretised image formation models

The Physics of Imaging – Image formation model

Direct/Forward model

$$\mathbf{y} = P(SHu)$$

- $u \in \mathcal{U}(\mathbb{R}^d)$ is the imaged (*continuous*) object,
- $\mathbf{y} \in \mathbb{R}^M$ is the vector of (*discrete*) measurements,
- $P : \mathbb{R}^M \to \mathbb{R}^M$ is a degradation operator (e.g., quantisation, noise),
- $H : \mathcal{U}(\mathbb{R}^d) \to \mathcal{Y}(\mathbb{R}^{d'})$ models the physics between the object and the signal to be recorded by the detector,
- $S : \mathcal{Y}(\mathbb{R}^{d'}) \to \mathbb{R}^M$ models the integration on the detector.

Direct/Forward model

$$\mathbf{y} = P(SHu)$$

Definitions of H and S

H and S are assumed to be **linear integral** operators,

$$\begin{aligned} H : \mathcal{U}(\mathbb{R}^d) &\longrightarrow \mathcal{Y}(\mathbb{R}^{d'}) \\ u &\longmapsto y(\,\cdot\,) = \int_{\mathbb{R}^d} h(\,\cdot\,, \mathbf{x}) u(\mathbf{x}) \,\mathrm{d}\mathbf{x}, \end{aligned}$$

where $h : \mathbb{R}^{d'} \times \mathbb{R}^d \to \mathbb{R}$ denote the **kernel** of H, and

$$\begin{aligned} S : \mathcal{Y}(\mathbb{R}^{d'}) &\longrightarrow \mathbb{R}^M \\ v &\longmapsto \mathbf{y} = \left[\int_{\mathbb{R}^{d'}} \psi_m(\mathbf{z}) y(\mathbf{z}) \,\mathrm{d}\mathbf{z} \right]_{m=1}^{M}, \end{aligned}$$

where $\{\psi_m\}_{m \in [M]}$ is a family of **sampling functions**.

Table 1: Kernels of basic linear integral operators.

Operator	$h(\mathbf{z}, \mathbf{x})$
Pointwise multiplication with function w	
Convolution with kernel $k(\cdot - \cdot)$	
Fourier transform	

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Physical phenomena $\approx$ compositions of basic linear operators

The Physics of Imaging – Blur

From diffraction to point spread function (PSF)

The finite aperture of optical elements induces diffraction (light blur).

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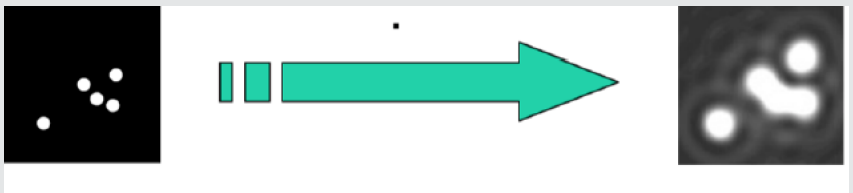
Limitations in image resolution



X-Ray transform

For $d \geq 2$, the X-ray transform is defined by the line integral

$$Hu(\mathbf{s}, \mathbf{w}) = \int_{L_{\mathbf{s},\mathbf{w}}} u \, \mathrm{d}l = \int_{\mathbb{R}} u(\mathbf{s} + t\mathbf{w}) \mathrm{d}t,$$

- for all compactly supported $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$
- $L_{\mathbf{s},\mathbf{w}} := \{\mathbf{x} = \mathbf{s} + t\mathbf{w}, \ \forall t \in \mathbb{R}\} \subset \mathbb{R}^d$: lines through $\mathbf{s} \in \mathbb{R}^d$ with direction $\mathbf{w} \in \mathbb{R}^d$.

Radon transform

For $d \geq 2$, the Radon transform is defined by the line integral

$$Ru(\boldsymbol{\theta}, \rho) = \int_{L_{\boldsymbol{\theta},\rho}} u \, \mathrm{d}\sigma = \int_{\mathbf{x} \cdot \boldsymbol{\theta} = \rho} u(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{x}),$$

- for all hyperplanes

  $$L_{\theta,\rho} := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \boldsymbol{\theta} = \rho\},$$

  with normal direction $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ and distance $\rho \in \mathbb{R}^+$ from the origin.
- In $d = 2$, this gives integration over lines with

  $$\boldsymbol{\theta} = (\cos\varphi, \sin\varphi), \qquad \boldsymbol{\theta} \perp \mathbf{w}.$$

The Physics of Imaging – X-ray transforms

Central to many imaging modalities

- Computed tomography (CT, PET) in medical imaging
- Electronic microscopy
- Radio astronomy
- ...

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Description within the general model

For $\mathbf{z} = (\mathbf{s}, \mathbf{w}) \in \mathbb{R}^d \times \mathbb{R}^d$, $\mathbf{x} \in \mathbb{R}^d$

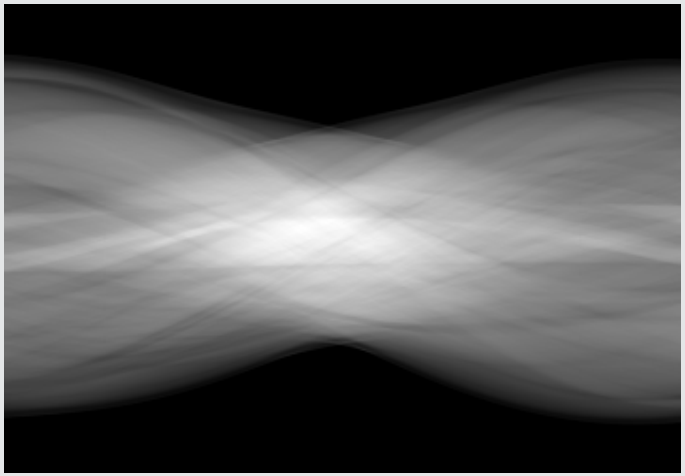
$$h(\mathbf{z}, \mathbf{x}) = h\left((\mathbf{s}, \mathbf{w}), \mathbf{x}\right) = \delta\left(\|\mathbf{P}_{\mathbf{w}^\perp}(\mathbf{x} - \mathbf{s})\|_2\right),$$

- $\mathbf{P}_{\mathbf{w}^\perp} \in \mathbb{R}^{(d-1)\times d}$ is the projection onto the subspace orthogonal to $\mathbf{w}$.

Link with Radon transform

In dimension $d = 2$, X-ray transform $\equiv$ Radon transform.

Object u



Sinogram $\mathbf{y}$ (Radon transform)

Different types of detector

- **photosensitive detectors**: in microscopy/tomography,
- **pressure sensors**: ultrasound imaging,
- **antennas**: magnetic resonance imaging (MRI, Fourier measurements).

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A "general" description of a detector

A finite number of **compact domains** $(A_m)_{m\in[M]}$, $A_m \subset \mathbb{R}^{d'}$ over which the received signal is *integrated* with $\psi_m = f_m \mathbb{1}_{A_m}$ for function f_m (often constant).

When A_m is sufficiently small, the approximation $\psi_m = \delta_{\mathbf{x}_m}$ can be used so that $\mathbf{y}[m] = Hu(\mathbf{x}_m)$ (sampling operator).

From continuous to discrete – Objectives

Generic direct model

$$\mathbf{y} = P(SHu)$$

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Discretized model

Define a **discrete representation** $\mathbf{u} \in \mathbb{R}^N$ ($N \in \mathbb{N}$) of $u \in \mathcal{U}(\mathbb{R}^d)$ (e.g., through sampling, basis expansion) and **matrix** $\mathbf{A} \in \mathbb{R}^{M \times N}$ such that

- the approximation error $Ad\mathbf{u} \approx SHu$
- fast algorithms can be used to compute $\mathbf{Au}$ without building $\mathbf{A}$.

From continuous to discrete – General principle

Approximation space for u

Let $\{\phi_n\}_{n\in[N]} \subset \mathcal{U}(\mathbb{R}^d)$ be a family of linearly independent functions of $\mathcal{U}(\mathbb{R}^d)$.
Define:

$$\mathcal{U}_N(\mathbb{R}^d) = \mathrm{span}(\{\phi_n\}_{n\in[N]}) = \left\{ \mathbf{\Phi u} : \mathbf{u} \in \mathbb{R}^N \right\}$$

where $\mathbf{\Phi} : \mathbb{R}^N \longrightarrow \mathcal{U}(\mathbb{R}^d)$ is defined by

$$\mathbf{\Phi u} = \sum_{n\in[N]} u_n \phi_n.$$

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Finite-dimensional projection

Consider then $P_{\mathcal{U}_N}(u)$, the orthogonal projection of $u \in \mathcal{U}(\mathbb{R}^d)$ on $\mathcal{U}_N(\mathbb{R}^d)$

$$\mathbf{u} = P_{\mathcal{U}_N}(u) = \arg \min_{\mathbf{v}\in\mathcal{U}_N(\mathbb{R}^d)} \|\mathbf{v} - u\|^2_{\mathcal{U}(\mathbb{R}^d)}.$$

From continuous to discrete – General principle

Usual choice of the family $\{\phi_n\}_{n\in[N]}$

$$\phi_n = \phi(\cdot - \mathbf{x}_n)$$

- $\phi \in \mathcal{U}(\mathbb{R}^d)$ a **generating function** (e.g., sinc, spline, wavelet),
- $\Omega = \{\mathbf{x}_n\}_{n\in[N]}$ **grid** of N points of $\Omega \subset \mathbb{R}^d$ evenly spaced,
- such that the family is **linearly independent**.

From continuous to discrete – General principle

Discrete forward model

Let $u \in \mathcal{U}_N(\mathbb{R}^d)$ and its discrete representation $\mathbf{u} \in \mathbb{R}^N$, then $\forall m \in [M]$,

$$[SHu]_m = \sum_{n \in [N]} u_n [SH\phi_n]_m, \qquad \text{(apply only to elements } \phi_n\text{)}$$

Defining $[\mathbf{A}_0]_{m,n} = [SH\phi_n]_m$ we get

$$SHu = \mathbf{A}_0 \mathbf{u}$$

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Practical challenges

1. Coefficients $[\mathbf{A}_0]_{m,n}$ may not be computed analytically (depend on ϕ_n)
2. $\mathbf{A}_0 \in \mathbb{R}^{M \times N}$ is too large to be **stored** and to perform $O(MN)$ **matrix-vector products** $\mathbf{A}_0\mathbf{u}$.

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   - 500 GFlops processor,
   - M = N = 10 millions (image 2D $\approx 3162 \times 3162$ or 3D $\approx 500 \times 500 \times 40$),
   - 200 seconds to perform a standard matrix-vector product,
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Alternative : numerical integration + structured approximation

From continuous to discrete – Numerical integration

Quadrature rule

$$[\mathbf{A}_0]_{m,n} = \int_{\mathbb{R}^{d'}} \int_{\mathbb{R}^d} \psi_m(\mathbf{z})\, h(\mathbf{z},\mathbf{x})\, \phi_n(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{z}$$

$$\approx [\widehat{\mathbf{A}}]_{m,n} = \sum_{p\in[P]} \sum_{q\in[Q]} w_{p,q} \psi_m(\mathbf{z}_p) h(\mathbf{z}_p, \mathbf{x}_q) \phi_n(\mathbf{x}_q),$$

where the nodes $\{\mathbf{z}_p\}_{p\in[P]}$ and $\{\mathbf{x}_q\}_{q\in[Q]}$, as well as the coefficients $\{w_{p,q}\}_{(p,q)\in[P]\times[Q]}$, depend on the desired accuracy and the used quadrature.

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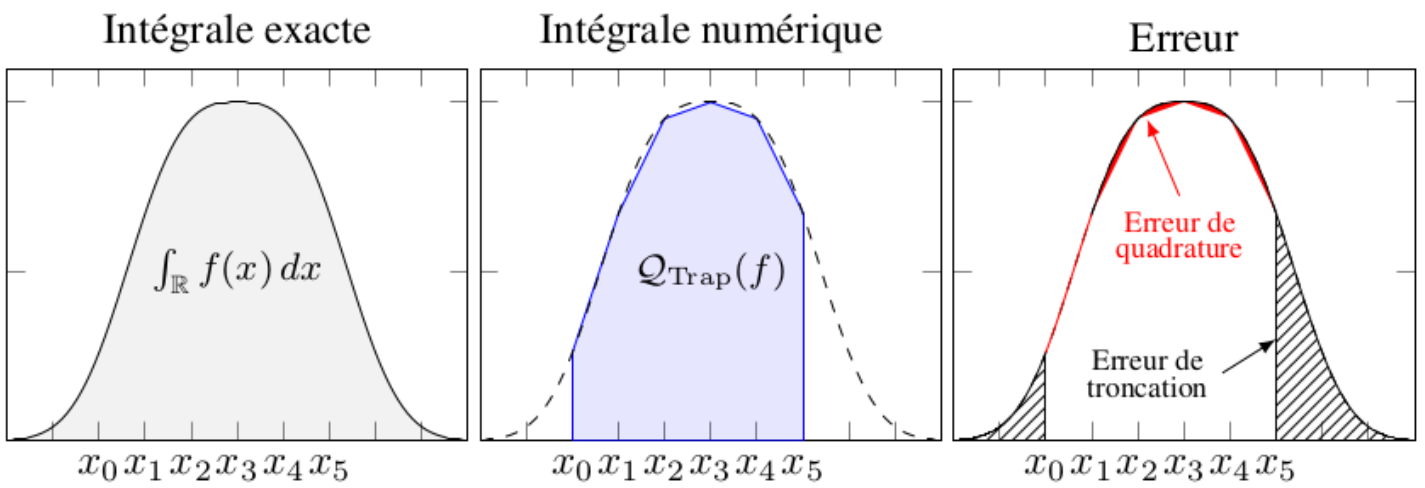

How to consider now **structured matrices A**? What is a structured matrix?

Sparse matrices

A matrix $\mathbf{B} \in \mathbb{R}^{M \times N}$ is **sparse** if it **contains many zeros**,

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \bullet & 0 & 0 & \bullet \\ 0 & \bullet & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & 0 & 0 & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

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From continuous to discrete – Structured matrices

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Memory and matrix-vector product complexities

If the K non-zeros are such that $K < (M(N-1)-1)/2$, the **Compressed Sparse Row** allows for

- memory/storage cost: $O(2K + M)$
- matrix vector product cost: $O(K)$

Sparse approximation

In practice $\widehat{\mathbf{A}}$ may not be sparse but **contains many small coefficients**
$\longrightarrow$ We can define a sparse approximation $\mathbf{A} = T_\epsilon(\widehat{\mathbf{A}})$ of $\widehat{\mathbf{A}}$ where

$$\forall \mathbf{B},\ \forall (m,n), \quad [T_\epsilon(\mathbf{B})]_{m,n} = \begin{cases} 0 & \text{ifi } |b_{m,n}| < \epsilon, \\ b_{m,n} & \text{otherwise.} \end{cases}$$

$\epsilon > 0$: **trade-off** between **approximation and computational complexity**.

Band/Multi-diagonal matrices

A square matrix $\mathbf{B} \in \mathbb{R}^{N \times N}$ is **multi-diagonal** if its non-zero entries are concentrated around the diagonal,

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Memory and matrix-vector product complexities

Only the L bands needs to be stored, leading to a memory and matrix vector product complexity of $O(LN)$.

Decaying kernels

Band matrices are well adapted to approximate matrices resulting from the discretization of integral operators with **decaying kernels** such that

$$h(\mathbf{z}, \mathbf{x}) \leq \frac{c}{\|\mathbf{z} - \mathbf{x}\|_2^{d+\alpha}}$$

for some $c > 0$ and $\alpha > 0$.
$\longrightarrow$ An approximation of $\widehat{\mathbf{A}}$ is given by $\mathbf{A} = T_\beta(\widehat{\mathbf{A}})$ with

$$\forall \mathbf{B}, \forall (m, n), \quad [T_\beta(\mathbf{B})]_{m,n} = \begin{cases} b_{m,n} & \text{if } |m - n| \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

$\beta > 0$: **trade-off** between **approximation and computational complexity**.

Circulant matrices

A square matrix $\mathbf{B} \in \mathbb{R}^{N \times N}$ is **circulant** if there exists $\mathbf{c} \in \mathbb{R}^N$ such that $b_{i,j} = c_{\mathrm{mod}(j-i,N)+1}$,

$$\mathbf{B} = \begin{pmatrix} c_1 & c_2 & \dots & c_{N-1} & c_N \\ c_N & c_1 & c_2 & \dots & c_{N-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & \dots & c_{N-1} & c_N & c_1 \end{pmatrix}.$$

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Circulant matrices can be **diagonalised in the Fourier basis**

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Memory and matrix-vector product complexities

Exploiting fast Fourier transform algorithms (FFT), we have

- matrix vector product cost : $O(2N\log N + N)$

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Remarks

- Special case of **sparse matrices** (in the Fourier basis)
- Discretisation of **convolution** operators with **periodic boundary conditions**

From continuous to discrete – Structured matrices

Low rank matrices

A matrix $\mathbf{B} \in \mathbb{R}^{M \times N}$ is **low rank** if its rank R is such that $R \ll \min(M, N)$.

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Memory and matrix-vector product complexities

Exploiting the singular value decomposition (SVD), $\mathbf{B} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ (with $\mathbf{U} \in \mathbb{R}^{M \times R}$, $\mathbf{\Sigma} \in \mathbb{R}^{R \times R}$ et $\mathbf{V} \in \mathbb{R}^{N \times R}$), we have

- memory cost : $O(R(M + N))$
- matrix vector product cost : $O(R(M + N))$

Table 3: Processor 500 GFlops per second.

	Complexit Thorique		Ex. $M = N = 10^7$	
Structure	Time	Memory	Time	Memory
K-Sparse	$O(K)$	$O(2K + M)$ *(format CSR)*	0,2 ms	1.68 Go $(K = 10^{-6}MN)$
L-Diagonal	$O(LN)$	$O(LN)$	0,06 ms	240 Mo $(L = 3)$
Circulant	$O(2N \log N + N)$	$O(2N)$	0,3 ms	160 Mo
Toeplitz	$O(4N \log(2N) + 2N)$	$O(4N)$	0,6 ms	320 Mo
Low rank (R)	$O(R(M + N))$	$O(R(M + N))$	0,4 ms	1.6 Go $(R = 10)$
Prod. standard	$O(MN)$	$O(MN)$	200 s	800 To

Outline

Practical information

From continuous to discretised image formation models

Ill-posedness

Maximum likelihood, MAP and MMSE

Regularisation: from priors to discrete models

Ill-posedness

Deconvolution example (square matrix A)

$$\mathbf{y} = \mathbf{A}\mathbf{u} + \mathbf{n}$$

with $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_N)$ and $\mathbf{A} \in \mathbb{R}^{N \times N}$ a **convolution operator (periodic BC)** with kernel $\mathbf{h}$,

$$(\mathbf{A}\mathbf{u})[\mathbf{m}] = (\mathbf{h} \star \mathbf{u})[\mathbf{m}] = \sum_{\mathbf{l}} \mathbf{h}[\mathbf{m} - \mathbf{l}]\mathbf{u}[\mathbf{l}]$$

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Direct inversion

$$\mathbf{u}_* = \mathbf{A}^{-1}\mathbf{y} = \mathbf{F}^{-1}\mathbf{diag}(\mathbf{F}\mathbf{h})^{-1}\mathbf{F}\mathbf{y}.$$

- When $\mathbf{n} = \mathbf{0}$, we have $\mathbf{u}_* = \mathbf{A}^{-1}\mathbf{A}\mathbf{u} = \mathbf{u}$
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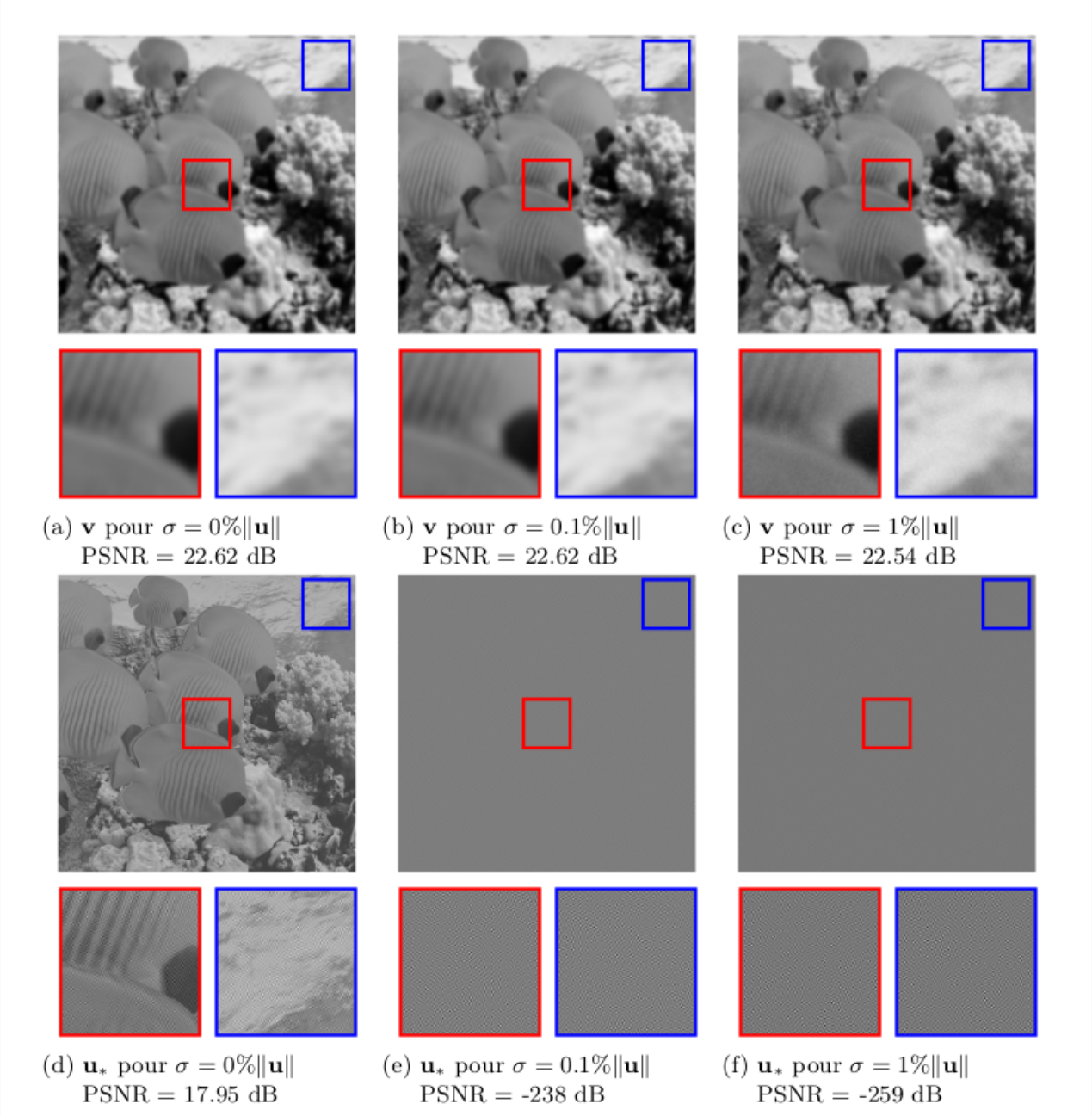
Ill-posed inverse problems – Failure of direct inversions



(a) $\mathbf{v}$ pour $\sigma = 0\%\|\mathbf{u}\|$
PSNR = 22.62 dB

(b) $\mathbf{v}$ pour $\sigma = 0.1\%\|\mathbf{u}\|$
PSNR = 22.62 dB

(c) $\mathbf{v}$ pour $\sigma = 1\%\|\mathbf{u}\|$
PSNR = 22.54 dB

(d) $\mathbf{u}_*$ pour $\sigma = 0\%\|\mathbf{u}\|$
PSNR = 17.95 dB

(e) $\mathbf{u}_*$ pour $\sigma = 0.1\%\|\mathbf{u}\|$
PSNR = -238 dB

(f) $\mathbf{u}_*$ pour $\sigma = 1\%\|\mathbf{u}\|$
PSNR = -259 dB

Ill-posed inverse problems – Definition

Well conditioned problem

A discrete inverse problem is **well conditioned** if a solution

- **exists** : $\mathbf{y} \in \mathrm{Im}(\mathbf{A})$
- is **unique** : $\mathbf{A}$ injective (i.e., $\ker(\mathbf{A}) = \{\mathbf{0}\}$)
- is **stable** to **perturbations**: $\|\mathbf{y}^\delta - \mathbf{y}\| \leq \delta$, then $\|\mathbf{u}^\delta - \mathbf{u}\| \leq C(\delta)$

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If one of these three conditions fails $\longrightarrow$ **ill-conditioned problem**.

Least-squares solution

Alternative to non-existence (i.e., $\mathbf{y} \notin \mathrm{Im}(\mathbf{A})$)

$$\mathbf{u}_* \in \arg \min_{\mathbf{u} \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{A}\mathbf{u} - \mathbf{y}\|_2^2.$$

- When $\mathbf{y} \in \mathrm{Im}(\mathbf{A})$, we have $\mathbf{A}\mathbf{u}_* = \mathbf{y}$,
- The least-squares problem **always admits a solution**, whatever $\mathbf{A}$ (also rectangular, $\mathbf{A} \in \mathbb{R}^{M \times N}$) and $\mathbf{y}$
- But the solution **may not be unique** if $\ker(\mathbf{A}) \neq \{\mathbf{0}\}$, since for all $\mathbf{w} \in \ker(\mathbf{A}) \backslash \{\mathbf{0}\}$ and all $\alpha \in \mathbb{R}$, $(\mathbf{u}_* + \alpha \mathbf{w})$ **is solution**.

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Minimum norm solution

Let $\mathcal{S} \subseteq \mathbb{R}^N$ be the set of least-squares solutions (possibly many values). The **minimum norm solution** is given by

$$\mathbf{u}_* = \arg \min_{\mathbf{u} \in \mathcal{S}} \|\mathbf{u}\|_2,$$

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It can be expressed as $\mathbf{u}_* = \mathbf{A}^\dagger \mathbf{y}$, with $\mathbf{A}^\dagger$ **Moore-Penrose pseudo inverse** :

$$\mathbf{A}^\dagger = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^* \tag{1}$$

where $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ is a SVD of $\mathbf{A}$.

Note that here we consider the "compact" SVD

- $\mathbf{U} \in \mathbb{R}^{M \times R}$ (with $\mathbf{U}^*\mathbf{U} = \mathbf{I}_R$);
- $\mathbf{V} \in \mathbb{R}^{N \times R}$ (with $\mathbf{V}^*\mathbf{V} = \mathbf{I}_R$);
- $\mathbf{\Sigma} = \mathbf{diag}\left(\sigma_1(\mathbf{A}) \cdots \sigma_R(\mathbf{A})\right) \in \mathbb{R}_+^{R \times R}$ (with $\sigma_1(\mathbf{A}) \geq \cdots \geq \sigma_R(\mathbf{A}) > 0$).

Stability and conditioning

Let denote

- $\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{u}}$ the noiseless data,
- $\bar{\mathbf{u}}$ the associated minimal norm solution ($\bar{\mathbf{u}} = \mathbf{A}^{\dagger}\bar{\mathbf{y}}$),
- $\mathbf{y} = \mathbf{A}\bar{\mathbf{u}} + \mathbf{n}$ the noisy data.

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Then we have

$$\|\bar{\mathbf{y}}\| = \|\mathbf{A}\bar{\mathbf{u}}\| \leq \|\mathbf{A}\|\|\bar{\mathbf{u}}\|$$
$$\|\mathbf{u}_* - \bar{\mathbf{u}}\| = \|\mathbf{A}^\dagger \mathbf{y} - \mathbf{A}^\dagger \bar{\mathbf{y}}\| \leq \|\mathbf{A}^\dagger\|\|\mathbf{y} - \bar{\mathbf{y}}\| = \|\mathbf{A}^\dagger\|\|\mathbf{n}\|,$$

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From which we get

$$\frac{\|\mathbf{u}_* - \bar{\mathbf{u}}\|}{\|\bar{\mathbf{u}}\|} \leq \kappa(\mathbf{A}) \frac{\|\mathbf{n}\|}{\|\bar{\mathbf{y}}\|},$$

with $\kappa(\mathbf{A})$ the **condition number** of $\mathbf{A}$,

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^\dagger\| = \frac{\sigma_1(\mathbf{A})}{\sigma_R(\mathbf{A})},$$

which could be potentially $\gg 1$ if $\sigma_R(\mathbf{A})$ is small...

Outline

Practical information

From continuous to discretised image formation models

Ill-posedness

Maximum likelihood, MAP and MMSE

Regularisation: from priors to discrete models

Maximum likelihood, MAP and MMSE

Bayesian formulation

Bayesian formalism

- $\mathbf{u}$ is a realisation of a random vector $\mathbb{u}$ with distribution $\pi_{\mathbb{u}}$
- $\mathbf{y}$ is a realisation of the random vector $\mathbb{y} = \mathcal{P}(\mathbf{A}\mathbb{u})$
- **Posterior distribution**[a]: $\pi_{\mathbb{u}|\mathbb{y}}(\mathbf{u}|\mathbf{y}) \propto \pi_{\mathbb{y}|\mathbb{u}}(\mathbf{y}|\mathbf{u})\pi_{\mathbb{u}}(\mathbf{u})$

[a] Stuart, Taeb, and Sanz-Alonso, '23

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Maximum A Posteriori (MAP) estimator

$$\hat{\mathbf{u}}_{\mathrm{MAP}}(\mathbf{y}) \overset{\mathrm{def}}{=} \arg\max_{\mathbf{u}} \pi_{\mathbb{u}|\mathbb{y}}(\mathbf{u}|\mathbf{y})$$

$\longrightarrow$ *Best* **point** *estimator*

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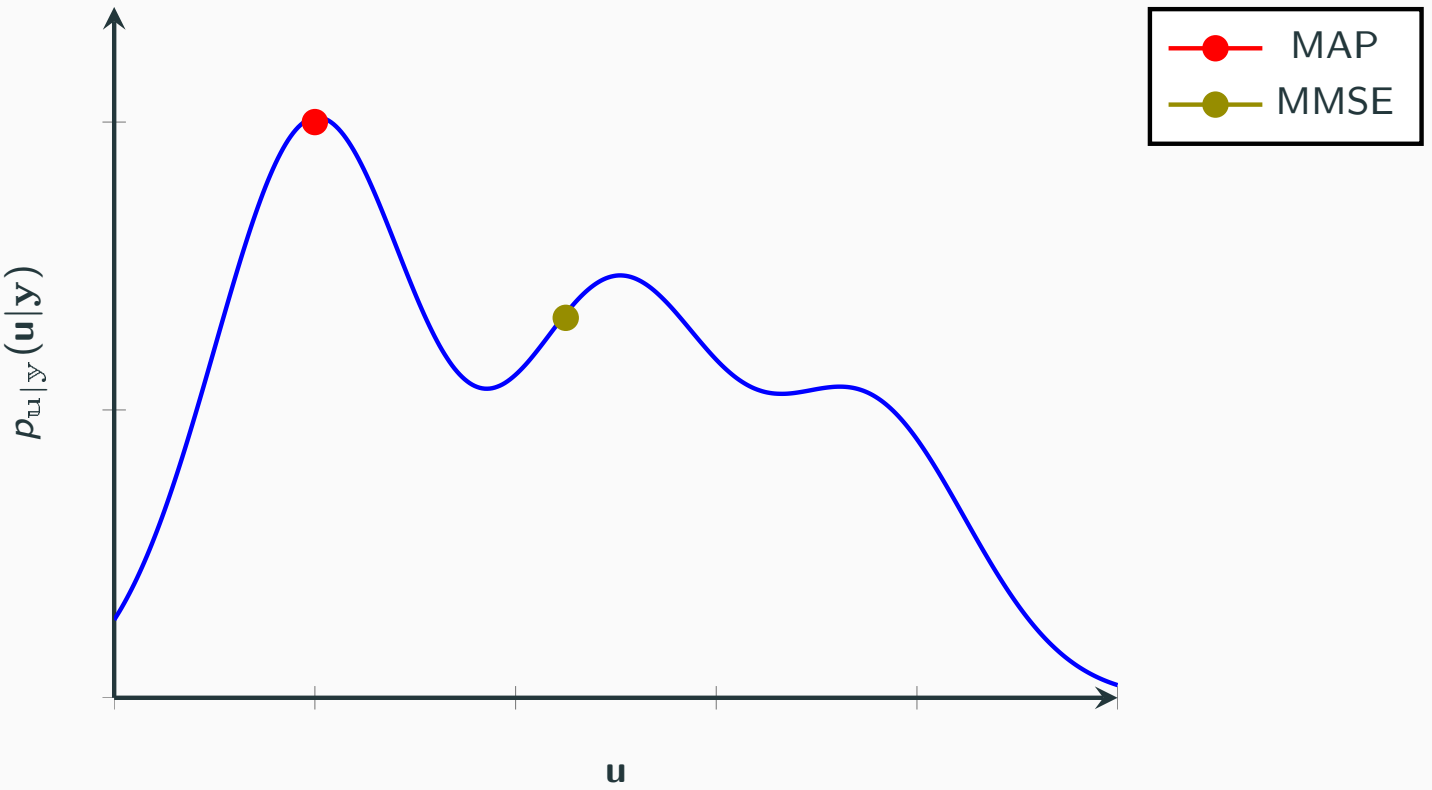
Minimum Mean Square Error (MMSE) estimator

$$\hat{\mathbf{u}}_{\mathrm{MMSE}} \overset{\mathrm{def}}{=} \arg\min_{\phi} \mathbb{E}_{(\mathbb{u},\mathbb{y})}\left[\|\phi(\mathbb{y}) - \mathbb{u}\|_2^2\right]$$

There holds: $\hat{\mathbf{u}}_{\mathrm{MMSE}}(\mathbf{y}) = \mathbb{E}\left[\mathbb{u}|\mathbb{y} = \mathbf{y}\right] = \int \mathbf{u}\pi_{\mathbb{u}|\mathbb{y}}(\mathbf{u}|\mathbf{y})\, d\mathbf{u}$

$\longrightarrow$ *Best estimator* **in average** *(according to posterior distribution)*

Bayesian formulation



Bayesian formulation – Parametric statistic viewpoint

Definitions and assumptions

- $\mathcal{P} \equiv P_\theta : \mathbb{R}^M \to \mathbb{R}^M$ is a **stochastic perturbation** parametrized by $\theta \in \mathbb{R}^s$
- $\mathbf{y}$ is a realization of a random vector with **law** $\pi_{\mathbb{y}}$ and parameters $(\theta, \mathbf{Au})$
- $\mathbf{u}$ is a realization of a random vector from the **a priori** law $\pi_{\mathbb{u}}$.

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White Gaussian noise	$\mathbf{y} = \mathbf{Au} + \mathbf{n}$	$\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma_\eta^2 \mathbf{I}_M)$	$\theta = \sigma_\eta$
Correlated Gaussian noise	$\mathbf{y} = \mathbf{Au} + \mathbf{n}$	$\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_\eta)$	$\theta = \mathbf{\Sigma}_\eta$
Poisson noise	$\mathbf{y} \sim \mathcal{P}(\mathbf{Au})$		
Poisson-Gaussian noise	$\mathbf{y} = \mathbf{z} + \mathbf{n}$	$\mathbf{z} \sim \mathcal{P}(\mathbf{Au})$ $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma_\eta^2 \mathbf{I})$	$\theta = \sigma_\eta$

Table 4: Examples of parametric stochastic perturbations P_θ

Bayesian formulation – Parametric statistic viewpoint

Maximum A Posteriori (MAP)

Maximize the probability of having $\mathbf{u}$ knowing $\mathbf{y}$ (posterior law),

$$\mathbf{u}_* \in \arg\max_{\mathbf{u}\in\mathbb{R}^N} \pi(\mathbf{u}|\mathbf{y})$$

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From Bayes formula $\pi(\mathbf{u}|\mathbf{y}) = \pi(\mathbf{y}|\mathbf{u})\pi(\mathbf{u})/\pi(\mathbf{y})$ and taking the negative log,

$$\mathbf{u}_* \in \arg\min_{\mathbf{u}\in\mathbb{R}^N} -\log\pi(\mathbf{y}|\mathbf{u}) - \log\pi(\mathbf{u})$$

- the **data fidelity** term $D(\mathbf{Au}, \mathbf{y}) \propto -\log\pi(\mathbf{y}|\mathbf{u})$ associated to the statistic of the perturbations
- the **regularization** term $R(\mathbf{u}) \propto -\log\pi(\mathbf{u})$ governed by the **a priori** distribution imposed on $\mathbf{u}$ (e.g., Gaussian law $\rightarrow$ quadratic)

Bayesian formulation – Examples of likelihood functions

Gaussian noise

$\mathcal{P}$ models an additive Gaussian noise $\mathbf{y} = \mathbf{A}\mathbf{u} + \mathbf{n}$ where $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$.

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$$\pi(\mathbf{y}|\mathbf{u}) = \frac{1}{(2\pi)^{M/2}\det(\mathbf{\Sigma})^{1/2}} \mathrm{e}^{-\frac{1}{2}(\mathbf{y}-\mathbf{A}\mathbf{u})^T\mathbf{\Sigma}^{-1}(\mathbf{y}-\mathbf{A}\mathbf{u})}$$

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$$D(\mathbf{A}\mathbf{u}, \mathbf{y}) = \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{u}\|^2_{\boldsymbol{\Sigma}^{-1}}$$

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Case of **white Gaussian noise**, $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_M$ and $D(\mathbf{A}\mathbf{u}, \mathbf{y}) = \frac{1}{2\sigma^2}\|\mathbf{y} - \mathbf{A}\mathbf{u}\|_2^2$.

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Data fidelity term

$$D(\mathbf{A}\mathbf{u}, \mathbf{y}) = \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{u}\|^2_{\boldsymbol{\Sigma}^{-1}}$$

where $\|\cdot\|^2_{\mathbf{W}} = \langle \mathbf{W}\cdot, \cdot \rangle$.

Case of **white Gaussian noise**, $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_M$ and $D(\mathbf{A}\mathbf{u}, \mathbf{y}) = \frac{1}{2\sigma^2}\|\mathbf{y} - \mathbf{A}\mathbf{u}\|_2^2$.

Properties of $D(\mathbf{A}\cdot, \mathbf{y})$

- infinitely differentiable
- convex
- L-Lipschitz gradient with $L = \|\mathbf{A}^*\mathbf{A}\|$.

Bayesian formulation – Examples of likelihood functions

Poisson noise

$\mathcal{P}$ models a Poisson noise[a] (counting process) $\mathbf{y} \sim \mathcal{P}(\mathbf{Au})$ **for u s.t.** $\mathbf{Au} > 0$.

[a] Bertero, Boccacci, '98

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$$D(\mathbf{Au}, \mathbf{y}) = \mathrm{KL}(\mathbf{Au}, \mathbf{y}) = \sum_{m \in [M]} (\mathbf{Au})_m - y_m \log((\mathbf{Au})_m)$$

known as the **Kullback-Leibler** divergence: not Lipschitz-smooth in $\mathbf{0}$.

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Properties of $\mathrm{KL}(\mathbf{A} \cdot + \boldsymbol{\epsilon}, \mathbf{y})$ **on** $\mathbb{R}^N_{\geq 0}$

- infinitely differentiable
- convex
- L-Lipschitz gradient with $L = \|\mathbf{A}\|^2 \max_m y_m / \epsilon_m^2$

Poisson denoising

$$\mathbf{y} = \mathcal{P}(\alpha \mathbf{u} / \|\mathbf{u}\|_\infty)$$

where $\alpha > 0$ is an estimate of the expected number of photons.

Standard quality metrics:

$$\mathrm{MSE}(\mathbf{u}, \widehat{\mathbf{u}}) = \frac{1}{N} \sum_{n=1}^{N} (u_n - \widehat{u}_n)^2 , \qquad \in [0, +\infty) \downarrow$$

$$\mathrm{PSNR}(\mathbf{u}, \widehat{\mathbf{u}}) = 10 \log_{10} \left(\frac{u_{\max}^2}{\mathrm{MSE}(\mathbf{u}, \widehat{\mathbf{u}})} \right) , \qquad \in (-\infty, +\infty) \uparrow$$

$$\mathrm{SSIM}\,[1](\mathbf{u}, \widehat{\mathbf{u}}) = \frac{(2\mu_{\mathbf{u}}\mu_{\widehat{\mathbf{u}}} + C_1)(2\sigma_{\mathbf{u}\widehat{\mathbf{u}}} + C_2)}{(\mu_{\mathbf{u}}^2 + \mu_{\widehat{\mathbf{u}}}^2 + C_1)(\sigma_{\mathbf{u}}^2 + \sigma_{\widehat{\mathbf{u}}}^2 + C_2)}, \qquad \in [0, 1] \uparrow$$

[1] Wang, Bovik, Sheikh, Simoncelli, '01

Bayesian formulation – Numerical examples

Poisson denoising

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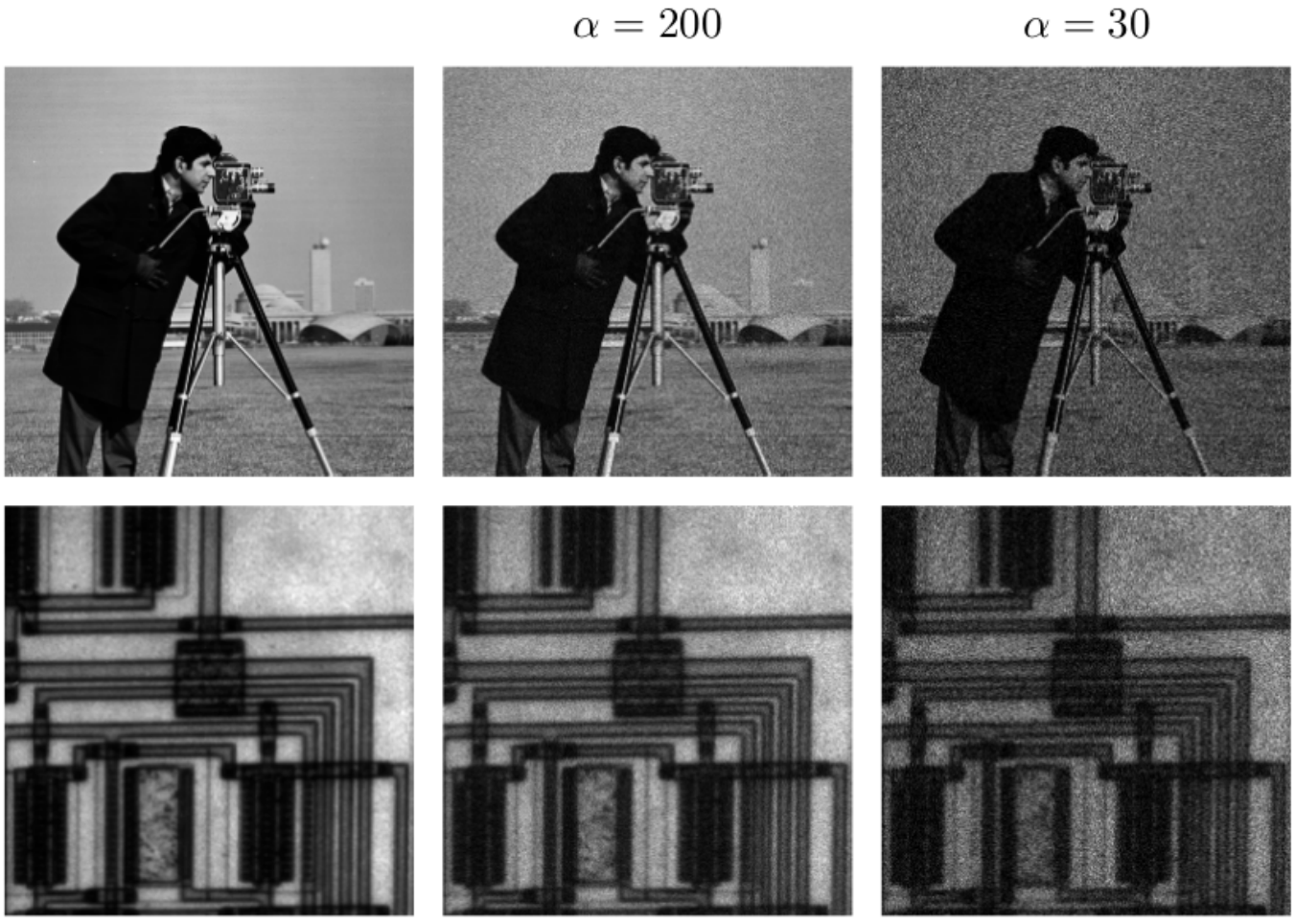

Bayesian formulation – Numerical examples

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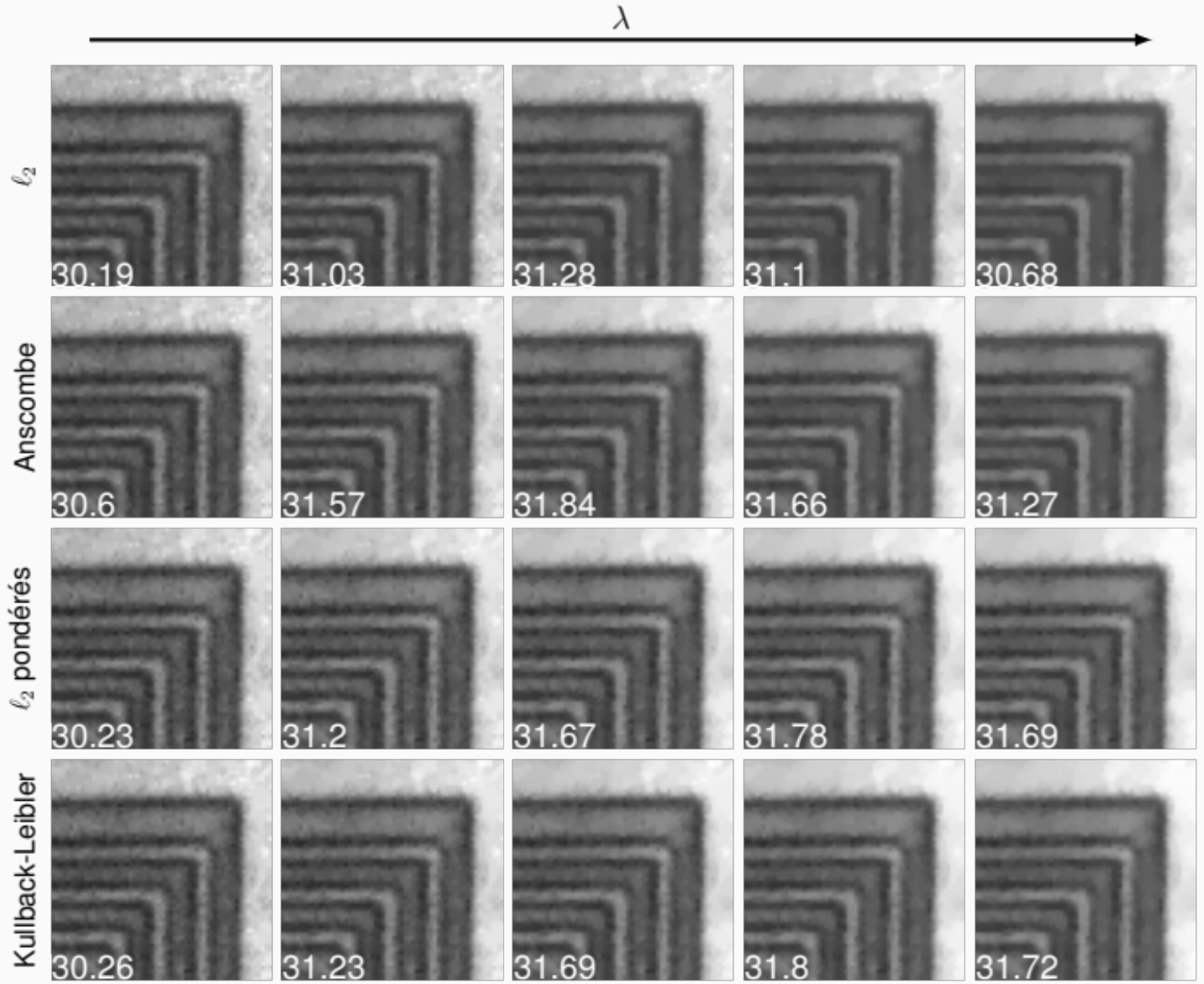

Bayesian formulation – Numerical examples

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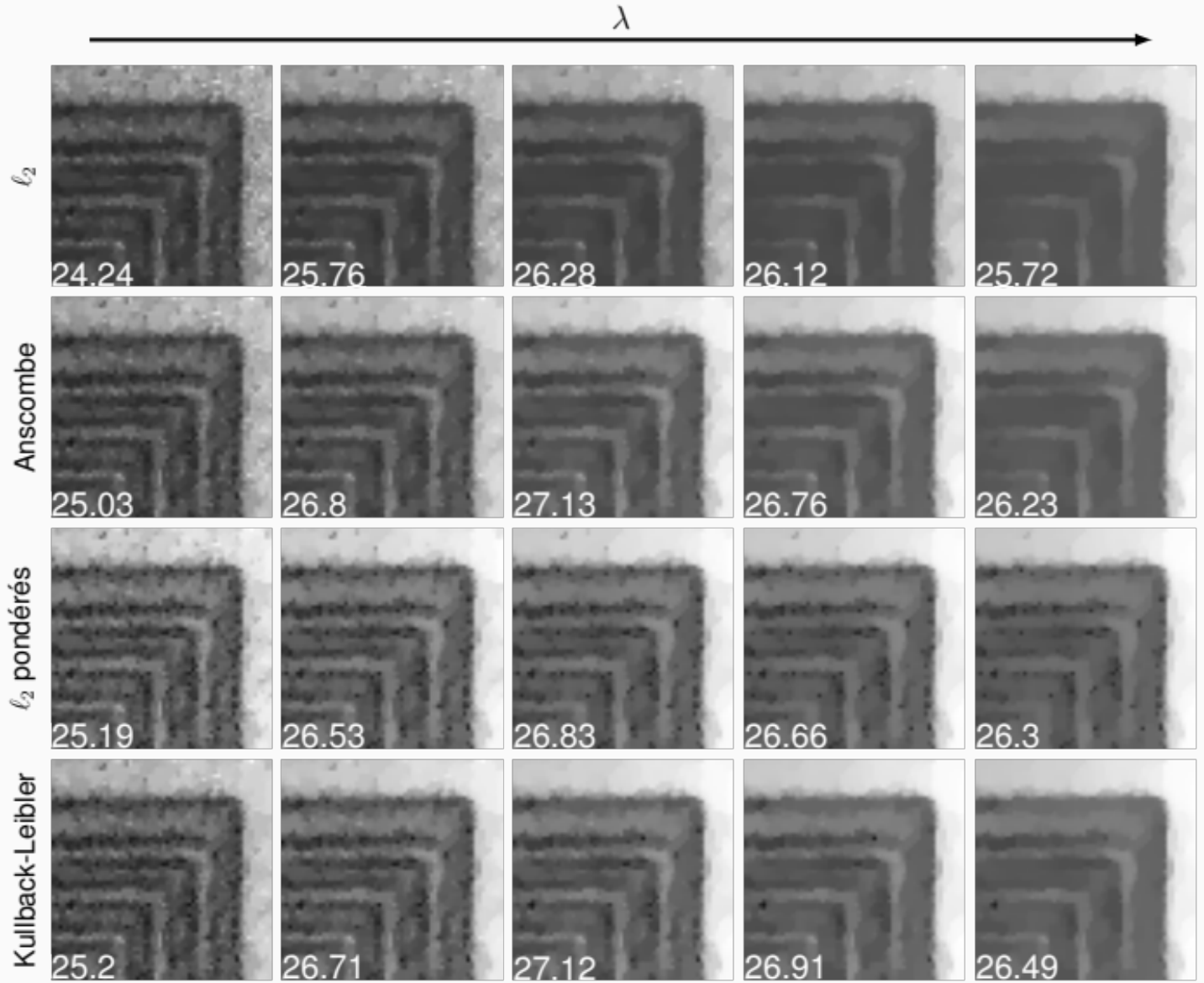


Outline

Practical information

From continuous to discretised image formation models

Ill-posedness

Maximum likelihood, MAP and MMSE

Regularisation: from priors to discrete models

Regularisation: from priors to discrete models

Regularisation idea– Why regularisation?

Inverse problems need prior information

Given measurements

$$\mathbf{y} \simeq SHu,$$

lack of injectivity: many objects may explain the same data.

Regularisation: among all plausible reconstructions, select the one that has desirable image features.

Generic variational model

$$u_* \in \arg\min_u D(SHu, \mathbf{y}) + \lambda R(u).$$

- $D(SHu, \mathbf{y})$: data fit with measurements.
- $R(u)$: prior knowledge on the image (smoothness, regularity...).
- $\lambda > 0$: tradeoff between data fidelity and prior.

Examples of priors

- small energy: $R(u) = \|u\|_2^2$,
- smoothness: $R(u) = \|\nabla u\|_2^2$,
- piecewise-constant structure: $R(u) = \mathrm{TV}(u)$,
- sparsity: $R(u) = \|u\|_1$.

Regularisation idea – Representer theorems (intuition)

Informal idea

A representer theorem says that, although the unknown image may live in a very large space, the regularised solution has a simple structure.

regularisation restricts the shape of admissible solutions.

Typical result

The solution has the following structure[a]:

$$u_* \approx \sum_{m=1}^{M} \alpha_m f_m.$$

- The coefficients α_m come from the data.
- The functions f_m depend on $\mathbf{y}$ and R

[a] Bredies, Carioni, '20

Why useful?

- It explains what kind of images a regulariser prefers.
- It suggests how to discretise the continuous model.

Regularisation idea – Structure of solutions for typical regularisers

Regulariser	Preferred structure
$\|u\|_2^2$	small-energy image
$\|\nabla u\|_2^2$	smooth image
$\mathrm{TV}(u)$	piecewise-smooth / edge-preserving image
$\|\alpha\|_1$	few non-zeros

Intuition

- Quadratic penalties spread energy smoothly.
- TV allows jumps and therefore preserves edges.
- ℓ^1 penalties select only a few active components.

Regularisation idea – Quadratic regularisation

Model

$$u_* \in \arg\min_u D(SHu, \mathbf{y}) + \frac{\lambda}{2}\|Lu\|_2^2,$$

with $L : \mathcal{U} \to \mathcal{U}'$. Typical choices are:

$$L = \mathbf{Id} \qquad \text{or} \qquad L = \nabla.$$

The resulting solution is usually smooth and globally distributed.

- $L = \mathbf{Id}$: penalises large image values.
- $L = \nabla$: penalises large variations.

Consequence

Quadratic regularisation tends to oversmooth sharp transitions, but it is convenient.

Regularisation idea – Discretised formulation

Discrete deconvolution problem

- $\mathbf{y} = \mathbf{A}\mathbf{u} + \mathbf{n}$
- $\mathbf{A} \in \mathbb{R}^{N \times N}$ convolution with kernel $\mathbf{h}$
- $\mathbf{n}$ Gaussian noise

Least-squares + quadratic (so-called Tikhonov/Sobolev) regularisation:

$$\mathbf{u}_* \in \arg \min_{\mathbf{u} \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{A}\mathbf{u} - \mathbf{y}\|_2^2 + \frac{\lambda}{2} \|\mathbf{D}\mathbf{u}\|_2^2.$$

Two basic choices

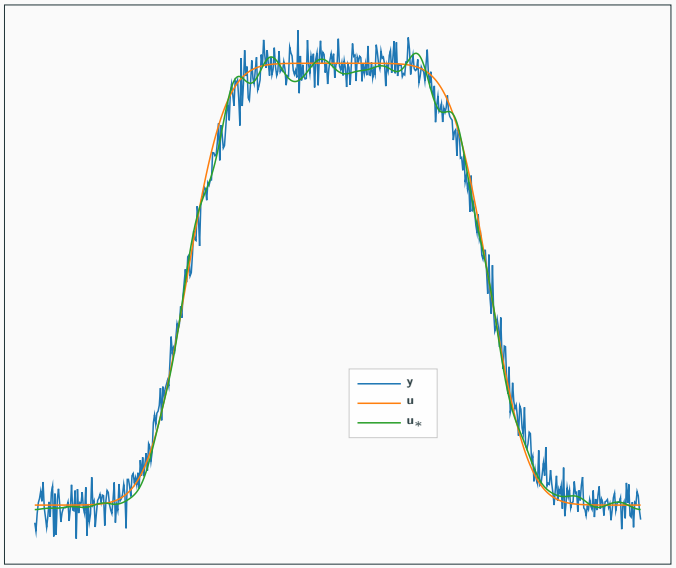
- $\mathbf{D} = \mathrm{Id}$: penalise the image itself.
- $\mathbf{D} = \nabla$: penalise finite differences, i.e. discrete gradients.

Regularisation idea – Tikhonov with $\mathbf{D} = \mathrm{Id}$

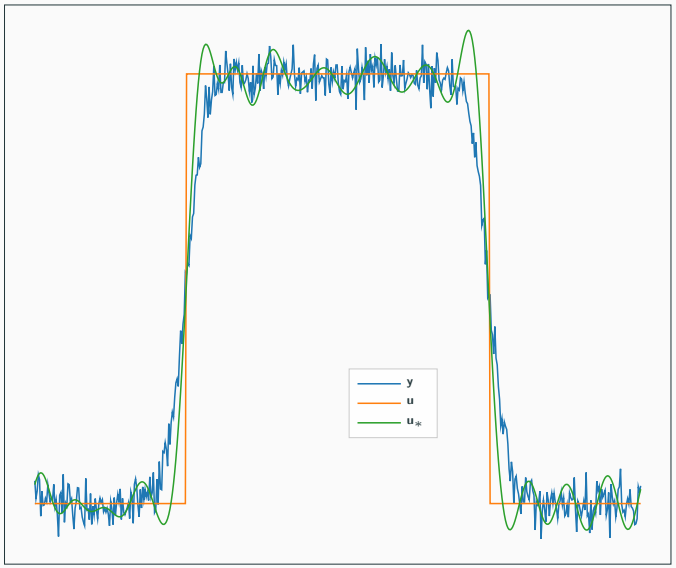
Interpretation

$$\mathbf{u}_* \in \arg\min_{\mathbf{u}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{A}\mathbf{u} - \mathbf{y}\|_2^2 + \frac{\lambda}{2}\|\mathbf{u}\|_2^2.$$

- Keeps the reconstruction energy small.
- Stabilises the inversion.
- Does not explicitly preserve edges.

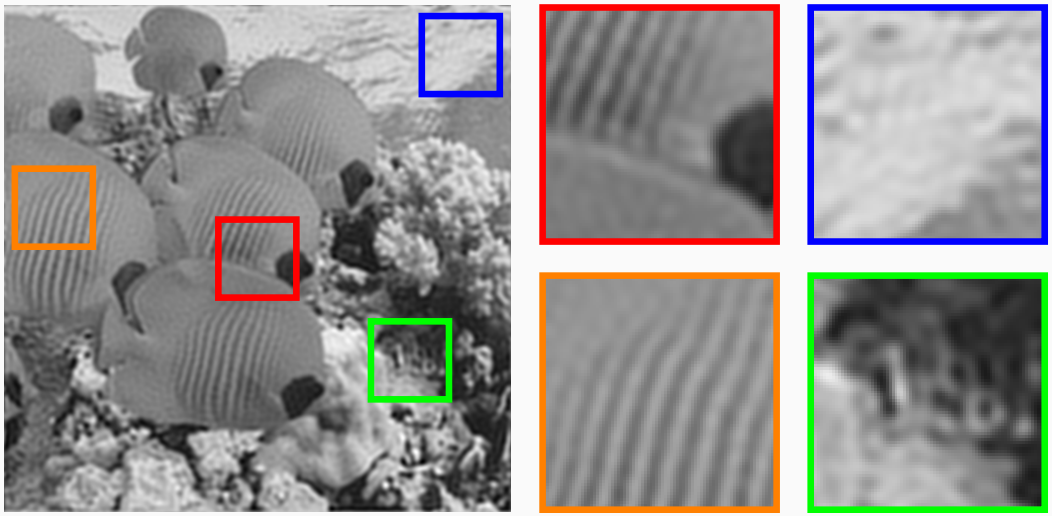


$\mathrm{PSNR}(\mathbf{u}_*) = 34.9\mathrm{dB}$; $\lambda = 0.01$



$\mathrm{PSNR}(\mathbf{u}_*) = 21.2\mathrm{dB}$; $\lambda = 0.005$

Regularisation idea – Tikhonov with $\mathbf{D} = \mathrm{Id}$: image example



$\mathbf{u}_*$ $(\mathbf{D} = \mathbf{Id})$ with $\sigma = 1\%$, PSNR $= 24.7$ dB and $\lambda = 10^{-2}$.

The reconstruction is stabilised, but sharp details may be smoothed.

Regularisation idea – Tikhonov/Sobolev regularisation, $\mathsf{D} = \nabla$

Gradient regularisation

$$\mathbf{u}_* \in \arg\min_{\mathbf{u}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{A}\mathbf{u} - \mathbf{y}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{\nabla}\mathbf{u}\|_2^2.$$

- Penalises oscillations.
- Encourages smooth reconstructions.
- Strong discontinuities are still penalised.

Finite-difference discretisation

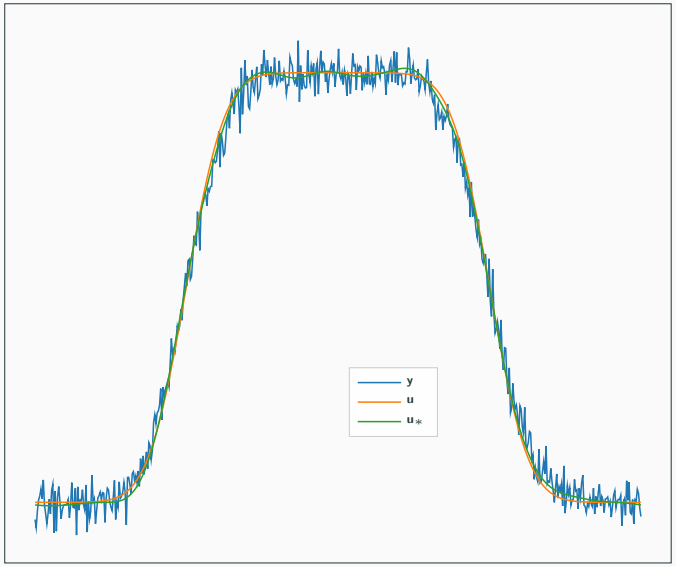
For a pixel grid,

$$\partial_1 u(\mathbf{x_n}) \simeq \mathbf{u}[\mathbf{n} + \mathbf{e}_1] - \mathbf{u}[\mathbf{n}] = \partial_1 \mathbf{u}[\mathbf{n}].$$

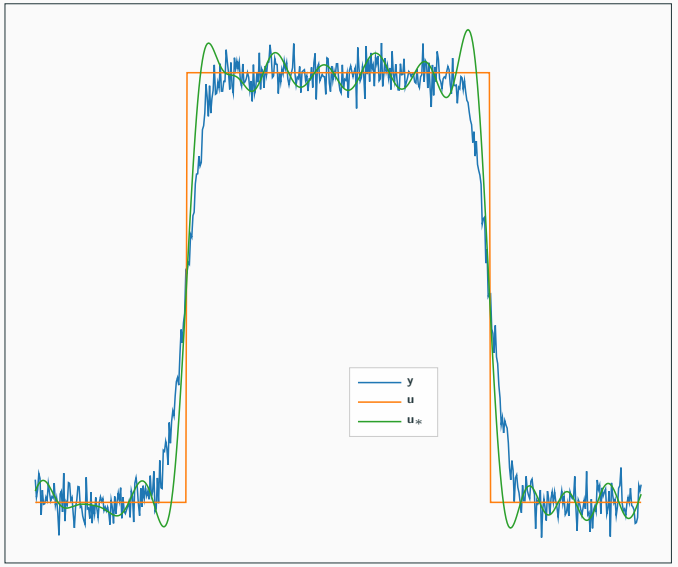
Thus,

$$\|\nabla u\|_2^2 \quad \rightsquigarrow \quad \sum_{\mathbf{n}} \|\boldsymbol{\nabla}\mathbf{u}[\mathbf{n}]\|_2^2.$$

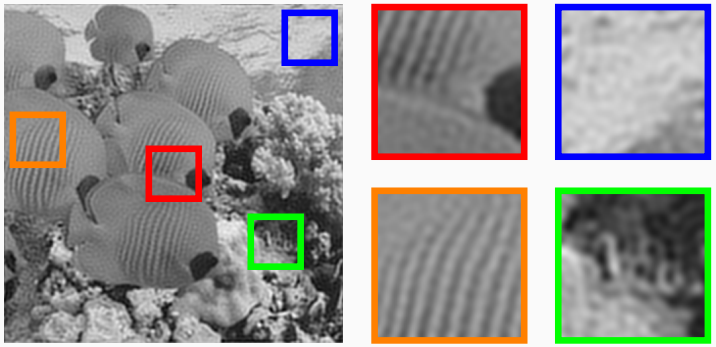
Regularisation idea – Tikhonov with $\mathbf{D} = \nabla$: examples



$\text{PSNR}(\mathbf{u}_*) = 40.7\text{dB}$; $\lambda = 10$



$\text{PSNR}(\mathbf{u}_*) = 21.3\text{dB}$; $\lambda = 0.3$



$\mathbf{u}_*$ $(\mathbf{D} = \boldsymbol{\nabla})$ with $\sigma = 1\%$, PSNR $= 24.6$ dB and $\lambda = 5.10^{-3}$.

Motivation

Natural images often contain:

- homogeneous regions,
- edges and contours.

Quadratic smoothness penalises edges too strongly.

TV regularisation

Instead of penalising squared gradients, TV penalises gradient magnitudes[a]:

$$\mathrm{TV}(\mathbf{u}) = \sum_{\mathbf{n}} \|\boldsymbol{\nabla}\mathbf{u}[\mathbf{n}]\|_2.$$

[a] Ruder, Osher, Fatemi, '92

Intuition

TV allows a few large gradients, hence it can preserve edges.

Total variation – TV representer

Preferred structure for TV solutions

$$\text{TV prior} \quad \Longrightarrow \quad \text{favours piecewise-constant images.}$$

- Flat regions are cheap.
- Edges are allowed.
- Too many oscillations are penalised.

Total variation – Drawback: staircasing effect

Remarks

- TV is edge-preserving.
- It promotes sparse gradients.
- It may transform smooth transitions into piecewise-constant plateaus.

Total variation – TV discretisation

Discrete image model

For a discrete image $\mathbf{u}$ represented on a pixel grid:

$$\mathbf{u} \in \mathbb{R}^N, \qquad N = N_1 \times N_2,$$

The discrete gradient is computed by finite differences:

$$\boldsymbol{\nabla} \mathbf{u}[\mathbf{n}] = (\partial_1 \mathbf{u}[\mathbf{n}], \partial_2 \mathbf{u}[\mathbf{n}]) \,.$$

Discrete total variation

$$\mathrm{TV}(\mathbf{u}) = \sum_{\mathbf{n} \in [N]} \|\boldsymbol{\nabla} \mathbf{u}[\mathbf{n}]\|_2 .$$

Total variation – TV deconvolution

In the deconvolution problem

- $\mathbf{y} = \mathbf{A}\mathbf{u} + \mathbf{n}$
- $\mathbf{A} \in \mathbb{R}^{N \times N}$ convolution with kernel $\mathbf{h}$
- $\mathbf{n}$ Gaussian noise

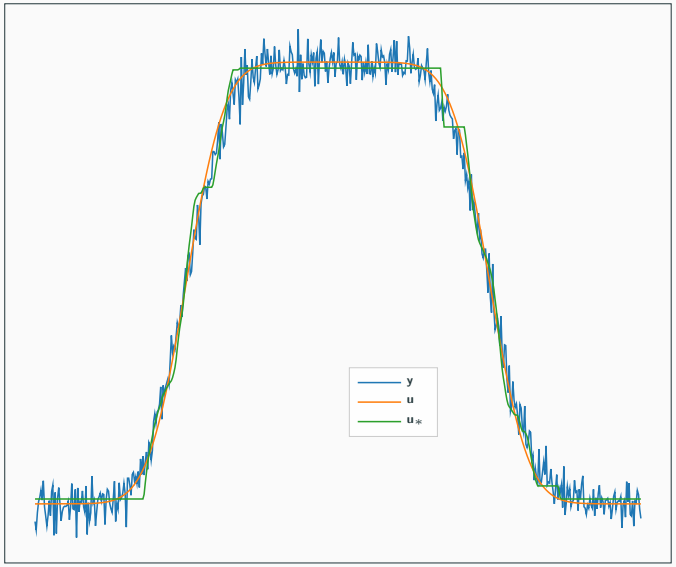
Least-squares TV:

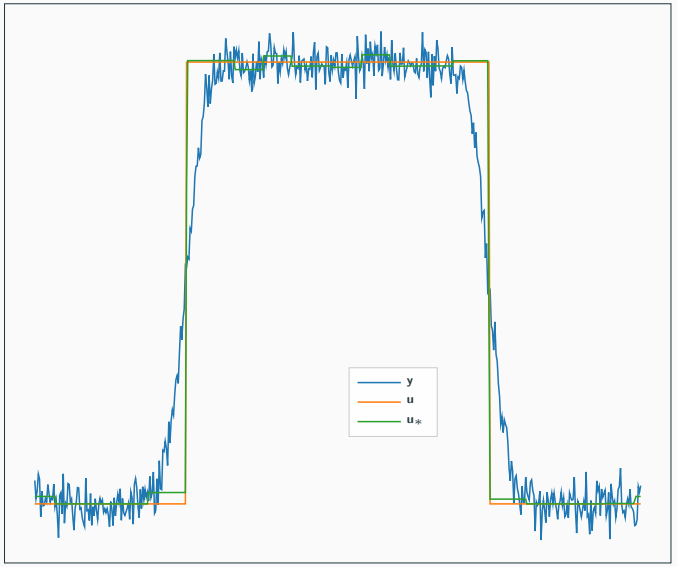
$$\mathbf{u}_* = \arg \min_{\mathbf{u} \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{A}\mathbf{u} - \mathbf{y}\|_2^2 + \lambda \mathrm{TV}(\mathbf{u}).$$

Take-away

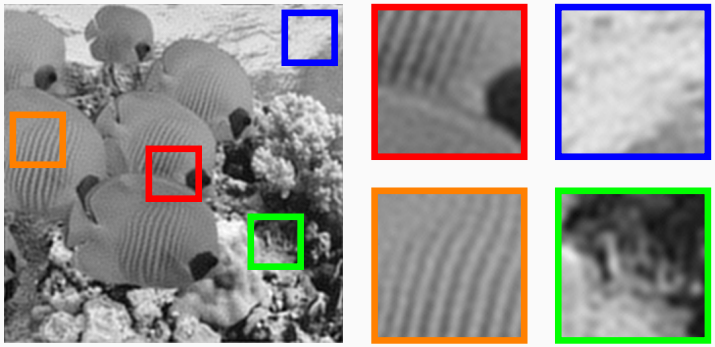
TV replaces the smoothness prior by an edge-preserving prior.

Total variation – TV deconvolution: examples





$\text{PSNR}(\mathbf{u}_*) = 32.3\text{dB}$; $\lambda = 0.67$ $\text{PSNR}(\mathbf{u}_*) = 30.2\text{dB}$; $\lambda = 0.5$



$\mathbf{u}_*$ $(L_2 - \mathrm{TV})$ with $\sigma = 1\%$, PSNR $= 25$ dB and $\lambda = 10^{-2}$.

Discrete gradient

For a discrete image $\mathbf{u} \in \mathbb{R}^{N_1 \times N_2}$, define the forward finite differences

$$(\nabla \mathbf{u})_{i,j} = \begin{bmatrix} u_{i+1,j} - u_{i,j} \\ u_{i,j+1} - u_{i,j} \end{bmatrix}.$$

Then

$$\nabla : \mathbb{R}^{N_1 \times N_2} \longrightarrow \left(\mathbb{R}^{N_1 \times N_2}\right)^2.$$

Total variation – Discretization of differential operators

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Discrete adjoint

Using discrete summation by parts,

$$\langle \nabla \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \nabla^* \mathbf{v} \rangle,$$

where

$$(\nabla^* \mathbf{v})_{i,j} = -(v^1_{i,j} - v^1_{i-1,j}) - (v^2_{i,j} - v^2_{i,j-1}).$$

Hence, the adjoint of the discrete gradient is the negative discrete divergence:

$$\nabla^* = -\operatorname{div}.$$

New viewpoint

Instead of constraining the image itself, we can constrain its representation in a suitable basis or dictionary. Assume the image can be represented as

$$u \simeq \sum_m \alpha[m]\psi_m.$$

- $(\psi_m)_m$: atoms (wavelets, Fourier, learned filters, ...)
- $\alpha[m]$: coefficients

Sparse representation

Natural images often admit such representations with only a few coefficients different from zero:

Sparse representation: idea

After discretisation, we write

$$\mathbf{u} \simeq \Psi\boldsymbol{\alpha}, \qquad \Psi \in \mathbb{R}^{N\times S}$$

where only a few coefficients of $\boldsymbol{\alpha} \in \mathbb{R}^S$ are non-zero.

- Ψ: dictionary, wavelet basis, learned basis, etc.
- $\boldsymbol{\alpha}$: coefficients.

Ideal sparse regularisation

$$\boldsymbol{\alpha}^* \in \arg\min_{\boldsymbol{\alpha}} D(\mathbf{A}\Psi\boldsymbol{\alpha}, \mathbf{y}) + \lambda\|\boldsymbol{\alpha}\|_0,$$

where the term counting the non-zero terms is:

$$\|\boldsymbol{\alpha}\|_0 = \#\{i : \boldsymbol{\alpha}_i \neq 0\}.$$

Problem

The ℓ^0 problem is combinatorial and generally hard to solve.

Sparsity – Relaxation of ℓ^0 problem

Convex relaxation

Replace the counting penalty by the ℓ^1 norm:

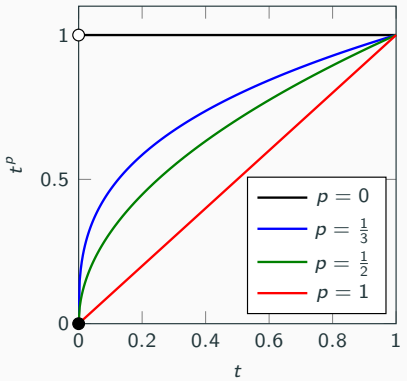
$$\|\boldsymbol{\alpha}\|_1 = \sum_i |\boldsymbol{\alpha}_i|.$$

The reconstruction problem becomes:

$$\boldsymbol{\alpha}_* \in \arg\min_{\boldsymbol{\alpha}} D(\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\alpha}, \mathbf{y}) + \lambda\|\boldsymbol{\alpha}\|_1.$$

Why ℓ^1?

- It promotes sparse coefficients.
- It is convex and computationally tractable.



Sparsity – ℓ^1 representer intuition

Informal representer intuition

The ℓ^1 penalty favours solutions defined in terms of a small number of atoms[a]:

$$\boldsymbol{\alpha}_* \text{ has many zero coefficients.}$$

Thus,

$$\mathbf{u}_* = \Psi \boldsymbol{\alpha}_*$$

is reconstructed from a few selected elements of the dictionary.

[a] Candés, Wakin, '08

Preferred structure

$$\ell^1 \text{ prior} \quad \Longrightarrow \quad \text{sparse representation.}$$

- The active atoms are selected from the data.
- The amplitudes are estimated by optimisation.
- The dictionary encodes the type of patterns we expect.

Sparsity – Geometry of sparse reconstruction

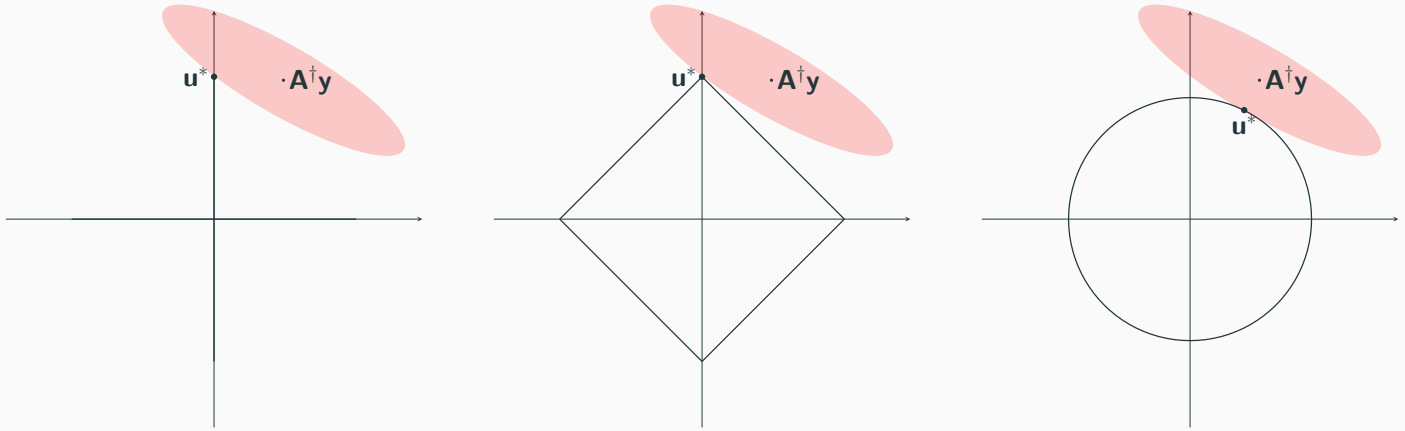
Convex proxy

The ℓ^1 problem is a useful proxy for the ideal ℓ^0 problem.

When does it work well?

This is studied in compressed sensing:

- exact recovery conditions,
- restricted isometry properties,
- sparse recovery guarantees.



ℓ_0 ℓ_1 ℓ_2

Sparsity – Summary: regularisers as image models

Main message

Regularisation is not only a numerical trick. It is an image model.

Regulariser	Promotes	Typical effect
$\|\mathbf{u}\|_2^2$	small energy	ringing
$\|\nabla \mathbf{u}\|_2^2$	smoothness	oversmoothing
$\mathrm{TV}(\mathbf{u})$	sparse gradients	good edges/staircasing
$\|\boldsymbol{\alpha}\|_1$	sparse coefficients	localised reconstructions

Bridge to algorithms

All these models lead to optimisation problems of the form

$$\min_{\mathbf{u}} D(\mathbf{A}\mathbf{u}, \mathbf{y}) + \lambda R(\mathbf{u}).$$

The next question is: how do we solve them efficiently?

Thank you!
Questions?

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