Computed Tomography in a nutshell

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What is Computed Tomography

Tomography: originates from the Greek words tomos (slice or section) and graphein (to write or record)

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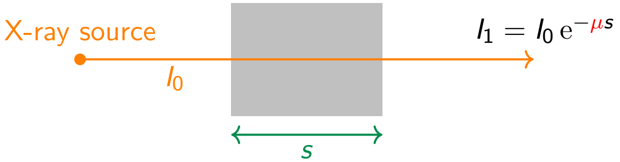
At the core:

- Measurements are taken exploiting the transmission of waves or particles (e.g., X-rays)
- The intensity of particles transmission is attenuated by the material through which they travel

Toy Example: A Line Inside Homogeneous Matter

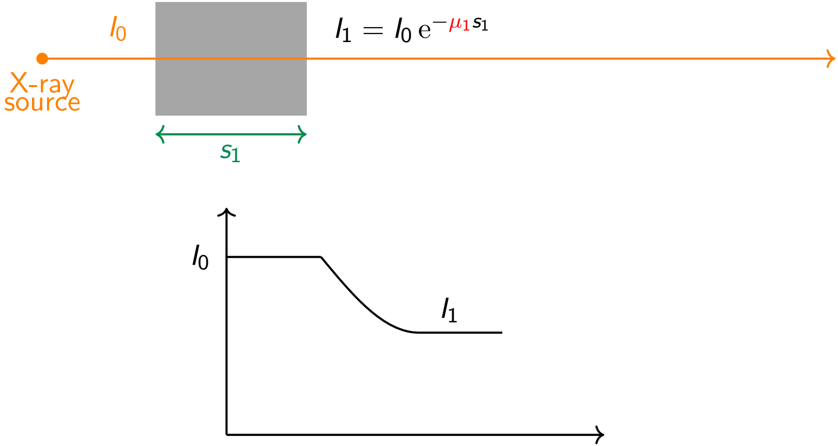


I_0: initial intensity of the X-ray

s: length of the path of the X-ray inside the object (particles are assumed to more or less travel in straight lines)

$\mu > 0$: X-ray attenuation coefficient

Toy Example: Two Homogeneous Blocks



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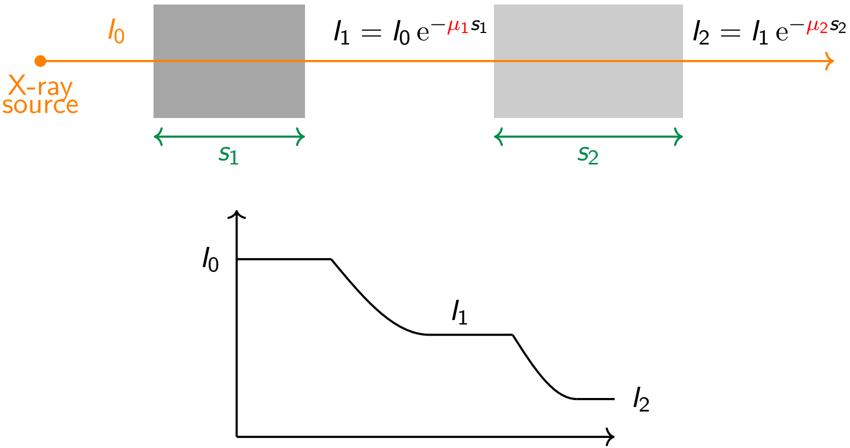
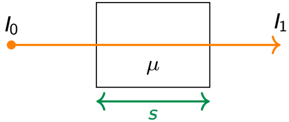


Absorption in the Target: the Beer-Lambert Law

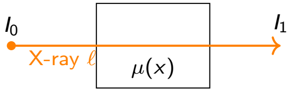
Homogeneous material:

$$I_1 = I_0 \ e^{-\mu s}$$



Non-homogeneous material:

$$I_1 = I_0 \ e^{-\int_\ell \mu(x)\,dx}$$



Absorption in the Target: Energy Dependence

Homogeneous material:

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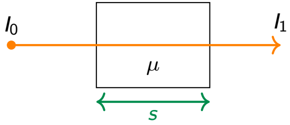

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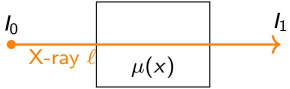

Transforming the Measurement for the Inverse Problem

The Beer-Lambert law connects the initial and final intensities of an X-ray:

$$I_1 = I_0 \; e^{-\int_\ell \mu(x)\, dx} \qquad \Longleftrightarrow \qquad -\log\left(\frac{I_1}{I_0}\right) = \int_\ell \mu(x)\, dx$$

where $-\log(I_1/I_0)$ models the total attenuated energy according the attenuation along ℓ.

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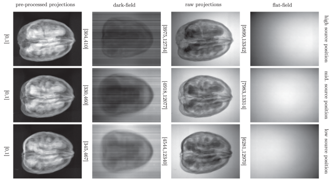
Before obtaining processed measurements, need to compensate "detector noise":

- **Dark-field** recorded with source off: detector offset count
- **Flat-field** with source on: the beam profile



[Der Sarkissian et al., Scientific Data, 2019]

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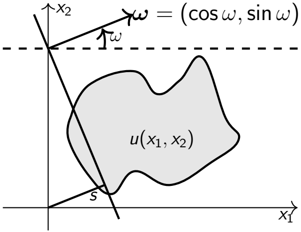
As a result, during a tomographic scan:

- I_0 is known from calibration and I_1 from measurements
- I_1 is measured along many lines $\ell_{(\omega,s)}$ to get many line integral values through the object
- The intensity I_1 is called the *transmission*, while the corresponding $-\log(I_1/I_0)$ is called absorption or **projection**, and a collection of projections is called a **sinogram**

Beer-Lambert Law and Radon Transform

The problem of recovering the attenuation function (linearised measurement) can be mathematically modelled by the **Radon transform**, which can be understood as an integration of the function $u : \mathbb{R}^2 \to \mathbb{R}_+$ over lines.

Through the identifications $u(\boldsymbol{x}) = \mu(\boldsymbol{x})$ and $\mathcal{R}(u) = -\log(I_1/I_0)$, the Beer-Lambert law is connected to the **Radon transform**:



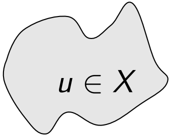
$$\mathcal{R}(u)(\boldsymbol{\omega}, s) = \int_{-\infty}^{\infty} u(s\boldsymbol{\omega} + \tau\boldsymbol{\omega}^{\perp})\, d\tau$$

$$= \int_{\ell(\boldsymbol{\omega},s)} u(\boldsymbol{x})\ d\boldsymbol{x},$$

where $\ell = \ell(\boldsymbol{\omega}, s) = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} = s\boldsymbol{\omega} + \tau\boldsymbol{\omega}^{\perp}, \tau \in \mathbb{R}\}$ with $\boldsymbol{\omega} = (\cos(\omega), \sin(\omega))$ and $(\boldsymbol{\omega}, s) \in S^1 \times \mathbb{R}$.

In Practice: How to Formulate CT as a Mathematical Problem

- We aim at imaging a target (e.g., human chest) $u \in X = L^2(\Omega)$ with $u : \mathbb{R}^d \to \mathbb{R}_+$ in a bounded domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$.



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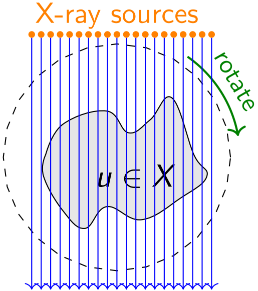
- The process of emitting X-rays that travel through target $u \in X$ is called the **forward problem/model** ($\rightsquigarrow$ Radon transform).



Forward Problem

$\mathcal{R}(u)$

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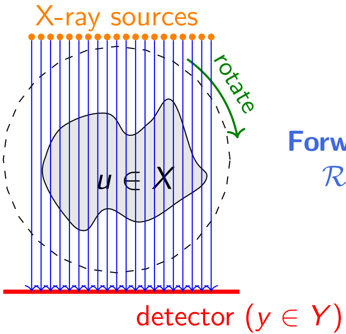

Forward Problem

$$\mathcal{R}(u) = y$$

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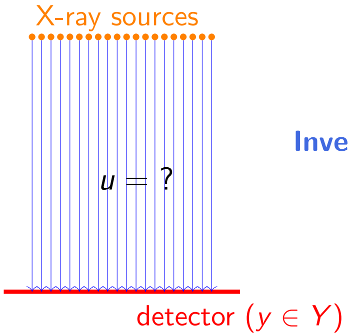
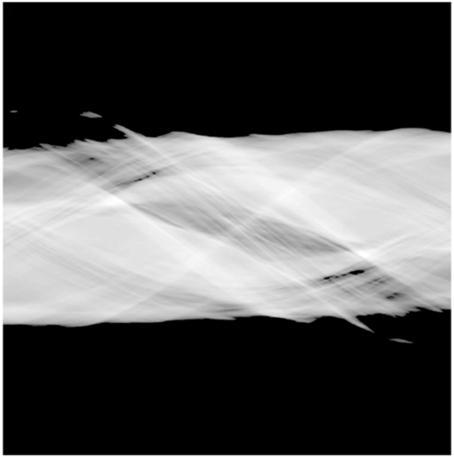
- Reconstructing f from the measured data y is then consequently the **inverse problem**.

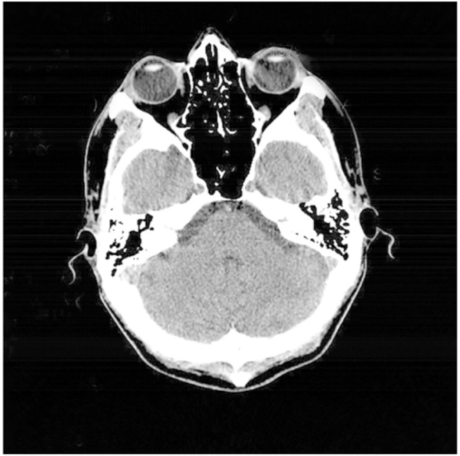


Inverse Problem

$u = ?$

Inversion formula: Filtered Backprojection (FBP)





Inversion formula: Filtered Backprojection (FBP)

Filtered Backprojection (FBP): inversion formula in the continuous case.

- ✓ Works well when comprehensive projection data are available
- ✓ Works well when the target is (assumed) static
- ✗ **Discretization**: theoretical limitations in the discrete setting.
- ✗ **Stability**: no stability guarantees $\rightarrow$ problems in real settings.

In many practical tomographic applications we wish to:

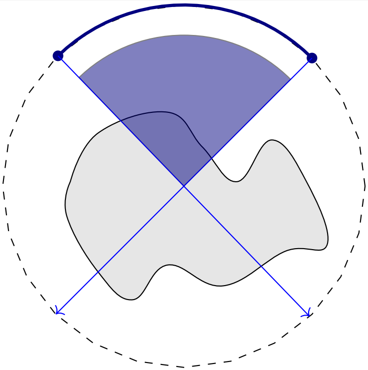
- lower the X-ray radiation dose
- shorten the scanning time

$\rightsquigarrow$ **Limited Data** tomography

Limited Data Tomography

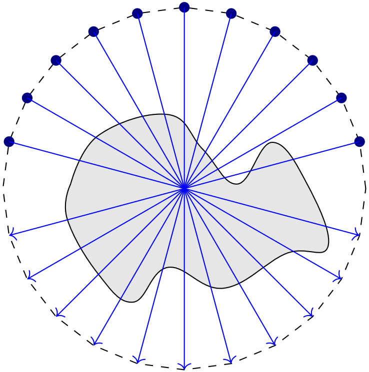
Limited-angle CT

We consider s equispaced angles in a restricted angular range.

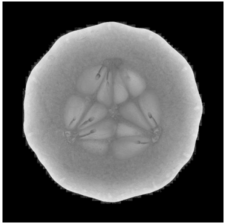


Sparse-angle CT

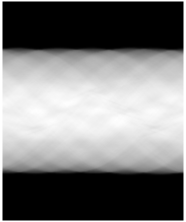
We consider s equispaced angles in the entire angular range.



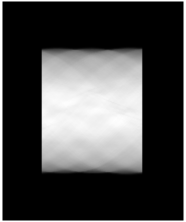
Limited Data Tomography



ground truth



Full angle data



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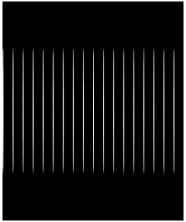

Sparse-angle data

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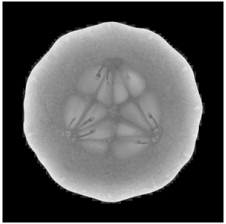

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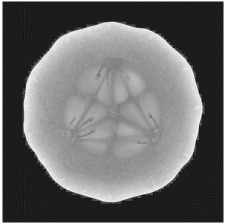
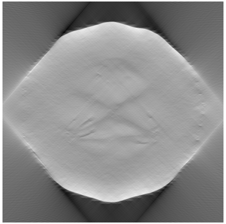
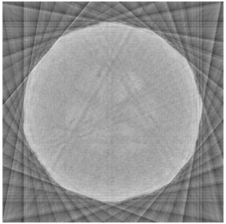


FBP from full angle data



FBP from limited-angle data



FBP from sparse-angle data