Recall: if $dX_t = b(X_t)dt + \sigma(X_t)dB_t$

with b, σ Lipschitz-continuous then

(on $\mathbb{R}$) $\quad Lf(x) = f'(x)b(x) + \frac{1}{2}f''(x)\sigma^2(x)$

(on $\mathbb{R}^n$) $\quad LF(x) = \nabla F(x)\cdot b(x) + \frac{1}{2}\mathrm{Tr}\left(F''(x)A\right)$, with $A = \sigma\cdot\sigma^*$

$$= \sum_{i=1}^{n} b^{(i)}(x)F_i'(x) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)F_{i,j}''(x)$$

Ex 2: Generator of Ornstein-Uhlenbeck process

Let $(X_t)_{t\geq 0}$ be the process on $\mathbb{R}$, solution of SDE

$$dX_t = -\lambda X_t dt + \sigma dB_t, \quad \text{with } \lambda, \sigma > 0 \quad \text{(Langevin equation)}$$

Then: $Lf(x) = -\lambda x f'(x) + \frac{\sigma^2}{2}f''(x)$

Solutions of SDE and PDE

$\boxed{A.}$ Dirichlet problem: look for solutions $h: \mathbb{R}^d \to \mathbb{R}$ s.t

$$(DP)\begin{cases} \Delta h(x) = 0, & \forall x \in D \\ h(x) = f(x), & \forall x \in \partial D \end{cases}, \quad \text{for } D \subseteq \mathbb{R}^d \text{ open, bounded set}$$

Remark: * Solutions should be $C^2(D)$ and $C(\bar{D})$

* Solutions are harmonic on D (w.r.t. to Δ)

Prop 1: If there exists a bounded solution of (DP), then it should be

$$\varphi(x) := \mathbb{E}_x\left(f(B_{\tau_{D^c}})\right), \quad x \in \mathbb{R}^d$$

where $\tau_{D^c} = \inf\{t > 0 : B_t \notin D\}$

Proof: First note that $\tau_{D^c} < \infty$, $\mathbb{P}_x$-a.s. ($(B_t - x)_{t\geq 0}$ is a standard BM)

* Let h be a bounded sol. of (DP) and consider $h(B_t)$, $t \geq 0$.

From the Ito's formula, $(h(B_{t\wedge\tau_{D^c}}))$ is a local martingale. Indeed

$$h(B_t) = h(B_0) + \int_0^t \nabla h(B_s)\cdot dB_s + \frac{1}{2}\int_0^t \underbrace{\Delta h(B_s)}_{=0} ds, \quad \text{for } t < \tau_{D^c}$$

$$= h(B_0) + \int_0^t \nabla h(B_s)\cdot dB_s \leftarrow \text{local martingale}$$

with $\mathbb{E}_x\left(h(B_{t\wedge\tau_{D^c}})\right) = \mathbb{E}_x\left(h(B_0)\right) = h(x) \quad (*)$

* Since $\begin{cases} \cdot\ t\wedge\tau_{D^c} \xrightarrow[t\to\infty]{} \tau_{D^c} & \mathbb{P}_x\text{-a.s} \\ \cdot\ h(B_{t\wedge\tau_{D^c}}) \xrightarrow[t\to\infty]{} h(B_{\tau_{D^c}}) & \mathbb{P}_x\text{-a.s.} \\ \cdot\ |h(B_{t\wedge\tau_{D^c}})| \le \sup_{y\in\bar{D}} |h(x)| < \infty \end{cases}$ (h and B are a.s continuous on ∂D) (h is bounded)

by dominated convergence, we get from (*)

$$\underline{h(x)} = \lim_{t\to\infty} \mathbb{E}_x\left(h(B_{t\wedge\tau_{D^c}})\right) = \mathbb{E}_x\left(f(B_{\tau_{D^c}})\right) \equiv \underline{\varphi(x)} \qquad \#$$

* Generalized Dirichlet problem

Let L be the generator of a Feller process (X_t) on $\mathbb{R}^d$.

We look for solutions $h: \mathbb{R}^d \to \mathbb{R}$ of the following

$$(\text{gen-DP}) \quad \begin{cases} Lh(x) = 0 & , x\in D\subset\mathbb{R}^d \text{ open, bounded set} \\ h(x) = f(x) & , x\in\partial D \end{cases}$$

Exercise: Prove that $\varphi(x) := \mathbb{E}_x\left(f(X_{\tau_{D^c}})\right)$ is the only possible bounded solution of (gen-DP).

Sugg: if h is a bounded sol., consider the martingale $(M_t^h)_{t\ge 0}$ ....

$\boxed{B}$ Poisson and Schrödinger equation

We look for solutions $u: \mathbb{R}^d \to \mathbb{R}$ s.t.

$$(\text{PE}) \quad \begin{cases} \frac{1}{2}\Delta u(x) = -g(x) & , \text{ if } x\in D\subset\mathbb{R}^d \text{ open, bounded} \\ u(x) = 0 & , \text{ if } x\in\partial D \end{cases}$$

$$(\text{SE}) \quad \begin{cases} \frac{1}{2}\Delta u(x) = -g(x)u(x) & , \text{ if } x\in D\subset\mathbb{R}^d \text{ open, bounded} \\ u(x) = f(x) & , \text{ if } x\in\partial D \end{cases}$$

Prop 2: Assume g bounded. If there exists a bounded solution of (PE), then it should be: $\varphi(x) := \mathbb{E}_x\left(\int_0^{\tau_{D^c}} g(B_t)\,dt\right)$, $x\in\mathbb{R}^d$.

Proof:

* Let u be a bounded sol. of (PE) and consider $M_t := u(B_t) + \int_0^t g(B_s)\,ds$

From the Ito's formula on $u(B_t)$, we get

$$M_t = u(B_0) + \int_0^t \nabla u(B_s)\cdot dB_s + \underbrace{\frac{1}{2}\int_0^t \Delta u(B_s)\,ds + \int_0^t g(B_s)\,ds}_{=0}$$

$$= u(B_0) + \int_0^t \nabla u(B_s)\cdot dB_s \qquad \leftarrow \text{local martingale for } t<\tau_{D^c}$$

with $\mathbb{E}_x(M_{t\wedge\tau_{D^c}}) = \mathbb{E}_x(M_0) = u(x)$ (*)

* Since $|\cdot\ t\wedge\tau_{D^c} \longrightarrow \tau_{D^c}$ $\mathbb{P}_x$-a.s

with $\mathbb{E}_x(M_{t\wedge T_{D^c}}) = \mathbb{E}_x(M_0) = u(x)$ $\quad (*)$

* Since $\begin{cases} \cdot\ t\wedge T_{D^c} \xrightarrow[t\to\infty]{} T_{D^c} \quad \mathbb{P}_x\text{-a.s} \\ \cdot\ u(B_{t\wedge T_{D^c}}) \xrightarrow[t\to\infty]{} u(B_{T_{D^c}}) = 0,\ \mathbb{P}_x\text{-a.s} \\ \cdot\ |u(B_{t\wedge T_{D^c}})| \le \sup_{y\in \bar{D}} |u(x)| < \infty \end{cases}$

(u and B are a.s continuous on ∂D)

(u is bounded)

by dominated convergence, we get from $(*)$

$$u(x) = \lim_{t\to\infty} \mathbb{E}_x(M_{t\wedge T_{D^c}}) = \underbrace{\mathbb{E}_x(u(B_{T_{D^c}}))}_{=0} + \mathbb{E}_x\left(\int_0^{T_{D^c}} g(B_s)\,dB_s\right) = \varphi(x)$$

\#

Prop 3: Assume g is bounded. If there exists a bounded solution of (SE) then it should be $\varphi(x) := \mathbb{E}_x\left(f(B_{T_{D^c}})\cdot \exp\left(\int_0^{T_{D^c}} g(B_s)\,ds\right)\right)$, $x\in\mathbb{R}^d$

Proof as an exercise!

Hint: Show that if u_t is a bounded sol. of (SE), then

$M_t = u(B_t)\cdot \exp\left(\int_0^t g(B_s)\,ds\right)$ is a local martingale for $t < T_{D^c}$

Then proceed as in the proofs of Prop 1 and Prop 2. \#

C The heat equation and the Feynman-Kac formula

We look for solutions $u: \mathbb{R}^+\times\mathbb{R}^d \to \mathbb{R}$ of

(HE) Heat equation $\begin{cases} \partial_t u = \frac{1}{2}\Delta u & \text{on } (0,+\infty)\times\mathbb{R}^d \\ (0,x) = f(x) \end{cases}$

($u(t,x)$ = temperature at point x at time t; $u(0,x)$ = initial temperature profile)

(FKE) Feynman-Kac equation $\begin{cases} \partial u = \frac{1}{2}\Delta u + g\cdot u & \text{on } (0,+\infty)\times\mathbb{R}^d \\ (0,x) = f(x) \end{cases}$

Prop 4: If there exists a bounded solution of (HE), then it should be

$$\varphi(t,x) := \mathbb{E}_x(f(B_t)) \quad , t\ge 0,\ x\in\mathbb{R}^d$$

Proof:

* Let u be a bounded sol. of (HE). Then, for all $t>0$ and all $s\in[0,t]$, let $M_t = u(t-s, B_s)$. Then $(M_s)_{s\in[0,t]}$ is a bounded martingale.

Indeed, by the Ito's formula:

$$M_s = M_0 + \int_0^s \nabla u(t-r, B_r)\,dB_r + \underbrace{\int_0^s -\partial_t u(t-r, B_r)\,dr + \frac{1}{2}\int_0^s \Delta u(t-r, B_r)\,dr}_{=0 \text{ by def of } u}$$

$$= u(t, B_0) + \int_0^s \nabla u(t-r, B_r)\,dB_r \leftarrow \text{(bounded martingale, as u bounded)}$$

Then $\mathbb{E}_x(M_t) = \mathbb{E}_x(M_0) = \mathbb{E}_x(u(t,B_0)) = u(t,x)$

$\| \quad \mathbb{E}_x(u(0,B_t)) = \mathbb{E}_x(f(B_t)) = \varphi(t,x)$ \#

Prop 5: Assume g is bounded. If there exists a solution of (FKE)

Prop. 5: Assume g is bounded. If there exists a solution of (FKE) that is bounded on $[0,T]\times\mathbb{R}^d$, $\forall T>0$, then it is given by

$$\varphi(t,x) := \mathbb{E}_x\left(f(B_t)\cdot \exp\left(\int_0^t g(t-r,B_r)\,dr\right)\right)$$

Proof:

* Let u be a bounded sol. of (FKE). For all $t>0$ and all $s\in[0,t]$, let $M_s := u(t-s,B_s)\cdot \exp\left(\int_0^s g(t-r,B_r)\,dr\right)$.

Then $(M_s)_{s\in[0,t]}$ is a bounded martingale.

$$M_s = M_0 + \int_0^s \nabla u(t-r,B_r)\exp(-)\,dB_r + \int_0^s \underline{-\partial_t u(t-r,B_r)\exp(-)\,dr}$$
$$+ \frac{1}{2}\int_0^s \underline{\Delta u(t-u,B_r)\exp(-)\,dr} + \int_0^s \underline{u(t-r,B_r)g(t-r,B_r)\exp(-)\,dr}$$

sum to $=0$, by def of u

$$= M_0 + \int_0^s \nabla u(t-r,B_r)\exp(-)\,dB_r .$$

Then $\mathbb{E}_x(M_t) = \mathbb{E}_x(M_0) = \mathbb{E}_x(u(t,B_0)) = u(t,x)$

$$\mathbb{E}_x(M_t) = \mathbb{E}_x\Big(\underbrace{u(0,B_t)}_{=f(B_t)}\exp\Big(\int_0^t g(t-r,B_r)\,dr\Big)\Big) = \varphi(t,x)$$

#

Lévy Processes

· Consider a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

Def: A process $(X_t)_{t\geq 0}$ defined on Ω and with value in $\mathbb{R}^d$ is said a Lévy process (LP) if it is $(\mathcal{F}_t)$-adapted and it satisfies:

* $X_0 = 0$ $\quad\mathbb{P}$-a.s
* X has càdlàg trajectories $\quad\mathbb{P}$-a.s
* $\forall s,t\geq 0$: $X_{t+s}-X_t \perp \mathcal{F}_t$, and $X_{t+s}-X_t \overset{d}{=} X_s$

$\Longleftrightarrow$ increments are independent and stationary

Examples

1. $X_t := bt$, $t\geq 0$ for given $b\in\mathbb{R}^d$ (deterministic, continuous traj.)
2. The BM in $\mathbb{R}^d$ is a Lévy process (continuous trajectories)

   $\longrightarrow$ $X_t := bt + cB_t$, $t\geq 0$ for given $b\in\mathbb{R}^d$, $c\in M_{d\times d}$
3. (Homogeneous) Poisson process $(N_t)_{t\geq 0}$ is a Lévy process on $\mathbb{R}^+$ (with càdlàg trajectories)
4. Compound Poisson process: $X_t = \sum^{N_t} Y_k$ $\quad t\geq 0$

(with càdlàg trajectories)

4. Compound Poisson process: $X_t = \sum_{k=1}^{N_t} Y_k$, $t \geq 0$
with $(N_t)_{t\geq 0}$ a $PP(\lambda)$, independent of $(Y_k)_{k\geq 0}$ iid r.v. on $\mathbb{R}^d$,
is a Levy process on $\mathbb{R}^d$ (with càdlàg trajectories)

We will see that in fact every LP on $\mathbb{R}^d$ is obtained by combining 1., 2., and 4., so to have

$$\text{“} X_t = \underbrace{bt}_{\text{drift}} + \underbrace{\sigma B_t}_{\text{diffusion}} + \underbrace{J_t}_{\text{jump process}} \text{”}$$

Properties (to start with): Let X a LP on $\mathbb{R}^d$

a. If $\mathcal{F}_t^X = \sigma(X_s, s \in [0,t])$, then X is a LP also w.r.t. $(\mathcal{F}_t^X)_{t\geq 0}$

b. If $c \in M_{m\times d} \Rightarrow c\cdot X$ is a LP on $\mathbb{R}^m$.

c. If X, Y are independent LP, then $X+Y$ is a LP

d. From b., each component $X^{(j)}$ of X, for $j=1,..,d$, is a LP (possibly dependent on the other components).

Lévy processes are intimately connected to infinite-divisible law, that we are going to define so to introduce the main tools to characterize LP.