Theorem [Strong Markov property]

Let $(X_t)_{t\geq 0}$ be a Feller process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and τ be $\mathbb{P}$-a.s. finite stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. Then, $\forall f: S \to \mathbb{R}$ measurable and bounded, and $\forall \nu \in \mathcal{P}(S)$,

$$(*) \qquad \mathbb{E}_\nu\left(f(X_{s+\tau}) \mid \mathcal{F}_\tau\right) = \mathbb{E}_{X_\tau}\left(f(X_s)\right) \qquad \mathbb{P}_\nu\text{-a.s}$$

In particular, the strong Markov property is equivalent to

$$\mathbb{P}_\nu\left(X_{s+\tau} \in A \mid \mathcal{F}_\tau\right) = \mathbb{P}_{X_\tau}\left(X_s \in A\right) \qquad \mathbb{P}_\nu\text{ - a.s}$$

$$\mathrm{Law}_\nu\left((X_{s+\tau})_{s\geq 0}\right) = \mathrm{Law}_{X_\tau}\left((X_s)_{s\geq 0}\right).$$

Proof: First, we prove it for τ taking value in a countable set, then move to the general setting considering a sequence of stopping times $(\tau_n)_{n\geq 0}$ s.t. $\tau_n \downarrow \tau$, and τ_n takes countable possible values.

1° STEP: Assume that τ takes values on $Q \subset [0,+\infty]$ countable

To prove $(*)$, we have to show that if $A \in \mathcal{F}_\tau$, then

$$\mathbb{E}_\nu\left[f(X_{s+\tau}) \cdot \mathbb{1}_A\right] \overset{?}{=} \mathbb{E}_\nu\left(\mathbb{E}_{X_\tau}(f(X_s)) \cdot \mathbb{1}_A\right)$$

$$\mathbb{E}_\nu\left[f(X_{s+\tau}) \cdot \mathbb{1}_A\right] = \sum_{q\in Q} \mathbb{E}_\nu\left(\underbrace{\mathbb{1}_{\{\tau=q\}} \cdot \mathbb{1}_A}_{\in \mathcal{F}_q} \cdot f(X_{s+q})\right)$$

$$= \sum_{q\in Q} \mathbb{E}_\nu\left[\mathbb{1}_{\{\tau=q\}} \cdot \mathbb{1}_A \, \underline{\mathbb{E}_\nu\left(f(X_{s+q}) \mid \mathcal{F}_q\right)}\right] \quad \text{Markov property}$$

$$= \sum_{q\in Q} \mathbb{E}_\nu\left[\mathbb{1}_{\{\tau=q\}} \cdot \mathbb{1}_A \, \mathbb{E}_{X_q}(f(X_s))\right] = \mathbb{E}_\nu\left[\mathbb{1}_A \cdot \mathbb{E}_{X_\tau}(f(X_s))\right] \quad \#$$

2° STEP: Let τ a general stopping time and consider

$$\tau_n := \begin{cases} 0 & \text{if } \tau = 0 \\ \frac{k+1}{2^n} & \text{if } \tau \in \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right] \end{cases}, \quad \forall n \in \mathbb{N}$$

so that τ_n take value on a countable set, $\tau_n \geq \tau$ and $\tau_n \searrow \tau$.

Then, for f bounded and continuous (hence in $C_0(S)$), and $A \in \mathcal{F}_\tau$

$$\mathbb{E}_\nu\left(f(X_{s+\tau_n}) \cdot \mathbb{1}_A\right) \overset{\text{previous step, with } A \in \mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}}{=} \mathbb{E}_\nu\left(\mathbb{E}_{X_{\tau_n}}(f(X_s)) \cdot \mathbb{1}_A\right) = \mathbb{E}_\nu\left(P_s f(X_{\tau_n}) \cdot \mathbb{1}_A\right)$$

$$\downarrow n \to \infty \qquad\qquad \downarrow n \to \infty$$

$$\mathbb{E}_\nu\left(f(X_{s+\tau}) \cdot \mathbb{1}_A\right) \qquad\qquad \mathbb{E}_\nu\left(P_s f(X_\tau) \cdot \mathbb{1}_A\right)$$

since $\begin{cases} \tau_n \searrow \tau \\ X \text{ is right-cont.} \\ f \text{ is cont} \end{cases}$ $\qquad$ since $\begin{cases} \tau_n \searrow \tau \\ X \text{ is right-cont} \\ P_s f \text{ is cont. (Feller property)} \end{cases}$

Since any bounded measurable function can be approximated by bounded continuous functions, we get the complete proof. #

The (Markov) Feller processes are indeed many, and also appear as solutions of SDE under proper conditions.

Theorem [Feller process and SDE] (without proof)
Let $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to M_{n \times d}$ be Lipschitz and bounded functions.
Then the (unique) solution of (c) is a Feller process.

$$(c) \begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \\ X_0 = x \end{cases}; \quad x \in \mathbb{R}^n, \ (B_t) \text{ BM on } \mathbb{R}^d$$

Generator

As for t-MC, we would like to characterize the law of a MP in a simpler way, as $(P_t)_{t\geq 0}$ is known explicitly only in few examples.

<u>Idea</u>: we would like to define L s.t. "$P_t = e^{tL}$", but in general L may not be bounded, and we have to specify the domain.

<u>Def</u>: Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$.
The generator of X is the operator $L : D_L \to C_0(S)$, $f \to Lf$ s.t.

$$Lf(x) := \lim_{t \downarrow 0} \frac{P_t f(x) - f(x)}{t}, \quad x \in S$$

where $D_L := \{ f \in C_0(S) \text{ s.t. } Lf \text{ is well defined} \}$
↳ the limit above exists

<u>Remark</u>: The generator describes the dynamics for infinitesimal intervals:

$$\underbrace{\mathbb{E}_\nu\left(f(X_{t+h}) - f(X_t) \mid \mathcal{F}_t\right)}_{\text{expected increment of } f(X_t)} = P_h f(X_t) - f(X_t) = h \cdot Lf(X_t) + o(h)$$

↑ Markov property

<u>FACT</u>: While $D_L \subsetneq C_0(S)$ in general, D_L is dense in $C_0(S)$.

We mention, without proof, some properties of the generator analogue to what we obtained in the countable setting (for t-MC):

<u>Theorem [Hille-Yosida]</u>: Let $f \in D_L$. Then

1. $P_t f \in D_L$, $\forall t \geq 0$

"e^{tL}"

2. $P_t f = \lim \left(Id - \frac{t}{m} L\right)^{-m} f$, $\forall f \in C_0(S)$ $\forall t$

1. $P_t f \in D_L$, $\forall t \geq 0$

2. $P_t f = \lim_{m \to \infty} \left(Id - \frac{t}{m} L \right)^{-m} f$, $\forall f \in C_0(S)$, $\forall t$

3. $\frac{d}{dt} P_t f = L P_t f = P_t L f$ (backward / forward Kolmogorov eq.s)

As we have done for MC, we can construct a family of martingales associated to Feller processes.

Theorem: Let $(X_t)_{t \geq 0}$ be a Feller process and $f \in D_L$.
Then the process $(M_t^f)_{t \geq 0}$, given by

$$M_t^f := f(X_t) - f(X_0) - \int_0^t L f(X_s)\, ds$$

is a $(\mathcal{F}_t)_{t \geq 0}$-martingale w.r.t. $\mathbb{P}_\nu$, $\forall \nu \in \mathcal{P}(S)$.

Proof: Notice that for $f \in D_L \subset C_0(S)$, $Lf \in C_0(S)$, thus one both bounded $\Rightarrow M_t^f \in L^1(\Omega, \mathcal{F}, \mathbb{P}_\nu)$. Moreover

$$\mathbb{E}_\nu \left(M_t^f - M_s^f \mid \mathcal{F}_s \right) = \mathbb{E}_\nu \left[f(X_t) - f(X_s) - \int_s^t L f(X_u)\, du \,\Big|\, \mathcal{F}_s \right]$$

Markov property

$$\overset{\downarrow}{=} \mathbb{E}_{X_s} \left[f(X_{t-s}) \right] - f(X_s) - \mathbb{E}_{X_s} \left[\int_0^{t-s} L f(X_u)\, du \right] \quad (\mathbb{P}_\nu\text{-a.s.})$$

$$= P_{t-s} f(X_s) - f(X_s) - \int_0^{t-s} P_u L f(X_s)\, du$$

Forward Kolmogorov eq.

$$= \int_0^{t-s} \left(\frac{d}{du} P_u f(X_s) - P_u L f(X_s) \right) du \overset{\downarrow}{=} 0$$

\#

Computation of the generator for solutions of SDE

Consider $X_t := x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s$, $\forall t \geq 0$

By definition: $Lf(x) = \lim_{t \to 0} \frac{1}{t} \left[\mathbb{E}_x(f(X_t)) - f(x) \right]$, $f \in D_L$.

By definition: $Lf(x) = \lim_{t \to 0} \frac{1}{t}\left[\mathbb{E}_x(f(X_t)) - f(x)\right]$, $f \in D_L$.

As a first step, we have to compute $\mathbb{E}_x(f(X_t))$.

Recall the Ito formula:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \quad \text{Ito process}$$

- on $\mathbb{R}$, for $f: \mathbb{R} \to \mathbb{R} \in C^2$,

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)\underbrace{d\langle X\rangle_t}_{= \sigma^2(X_t)dt}$$

- on $\mathbb{R}^m$, for $F: \mathbb{R}^m \to \mathbb{R} \in C^2$, with $B = (B^{(1)}, \dots, B^{(d)})$ BM on $\mathbb{R}^d$

with $\begin{cases} \nabla F = (F_i')_{i=1,\dots,m}, & F_i' := \frac{\partial}{\partial x_i}F \\ F'' = (F''_{ij})_{i,j=1,\dots,m}, & F''_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}F \end{cases}$

$$dF(X_t) = \nabla F(X_t) \cdot dX_t + \frac{1}{2}\mathrm{Tr}\left(F''(X_t) \cdot d\langle X, X\rangle_t\right)$$
$$= \sum_{i=1}^m F_i'(X_t) \cdot dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^m F''_{ij}(X_t)\, d\langle X^{(i)}, X^{(j)}\rangle_t$$

Hence: if $dX_t = b(X_t)dt + \sigma(X_t)dB_t$

with b, σ Lipschitz-continuous then

$$(\text{on } \mathbb{R}) \qquad Lf(x) = f'(x)b(x) + \frac{1}{2}f''(x)\sigma^2(x)$$

$$(\text{on } \mathbb{R}^m) \qquad LF(x) = \nabla F(x) \cdot b(x) + \frac{1}{2}\mathrm{Tr}\left(F''(x)A\right), \text{ with } A = \sigma \cdot \sigma^*$$

$$= \sum_{i=1}^m b^{(i)}(x)F_i'(x) + \frac{1}{2}\sum_{i,j=1}^m a_{ij}(x)F''_{ij}(x)$$

Ex 1: Generator of BM $(B_t)_{t\geq 0}$ on $\mathbb{R}^n$ is $L = \frac{1}{2}\Delta$.

Indeed: $dF(B_t) = \nabla F(B_t)\cdot dB_t + \frac{1}{2}\Delta F(B_t)dt$

$$\Rightarrow \mathbb{E}_x\big(F(B_t)\big) = F(x) + \frac{1}{2}\mathbb{E}_x\left(\int_0^t \Delta F(B_s)\,ds\right)$$

$$\Rightarrow LF(x) = \lim_{t\to 0}\frac{1}{t}\Big(\mathbb{E}_x\big(F(B_t)\big) - F(x)\Big)$$

$$= \frac{1}{2}\cdot\lim_{t\to 0}\frac{1}{t}\mathbb{E}_x\left(\int_0^t \Delta F(B_s)\,ds\right) = \frac{1}{2}\Delta F(x)$$