ADVANCED STOCHASTIC PROCESSES - 20° LECTURE

Markov Processes

- Let (S, d) a metric, complete, separable space.
  polish

  In most of the example $S = \mathbb{R}, \mathbb{R}^d$, with Euclidian metric.

- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space.

Def [Markov process]: A stochastic process $X = (X_t)_{t \geq 0} : \Omega \to S^{\mathbb{R}^+}$ is a Markov process if, $\forall s, t \geq 0$, $A \in \mathcal{B}(S)$

(1) $\mathbb{P}(X_{t+s} \in A \mid \mathcal{F}_s) = \mathbb{P}(X_{t+s} \in A \mid X_s)$ a.s. (Markov property)

Moreover, X is a homogeneous MP if

$$\mathbb{P}(X_{t+s} \in A \mid X_s = x) = \mathbb{P}(X_t \in A \mid X_0 = x) =: p_t(x, A)$$

where p is the corresponding transition probability

(we will also use the notation: $\mathbb{P}(X_{t+s} \in A \mid X_s) = p_t(X_s, A)$)

Properties of the transition probabilities

1. $\forall t \geq 0$, $\forall x \in S$: $p_t(x, \cdot) : \mathcal{B}(S) \to [0,1]$ is a probability measure
2. $\forall t \geq 0$, $\forall A \in \mathcal{B}(S)$: $p_t(\cdot, A) : S \to [0,1]$ is $\mathcal{B}(S)$-measurable
3. It holds, $\forall x \in S, A \in \mathcal{B}(S)$, $s, t \geq 0$

$$p_{t+s}(x, A) = \int_S p_t(x, dy)\, p_t(y, A)$$

(Chapman-Kolmogorov equation)

Law of the process

* **Notation**: $\mathbb{P}_\nu$ denotes the law of a $(X_t)_{t \geq 0}$ s.t. $X_0 \overset{d}{=} \nu$,

with $\nu \in \mathcal{P}(S)$, while $\mathbb{P}_x$ [illegible] the corresponding [illegible]

* <u>Notation</u>: $\mathbb{P}_\nu$ denotes the law of a $(X_t)_{t\geq 0}$ s.t. $X_0 \overset{d}{=} \nu$, with $\nu \in \mathcal{P}(S)$, while $\mathbb{E}_\nu$ denotes the corresponding average. For $\nu = \delta_x$, with $x \in S$, we simply wrigth $\mathbb{P}_x$ (and $\mathbb{E}_x$)

* <u>Finite dimensional distributions</u>

As the process takes value on the product space $S^{\mathbb{R}^+}$, its law is characterized by the finite-dimensional distributions "$\mathbb{P}(X_{t_0} \in dx_0, \dots, X_{t_m} \in dx_m)$", for $0 = t_0 < t_1 < \dots < t_m$, $\forall m \in \mathbb{N}$.

In particular, from the Markov property:

$$\mathbb{P}(X_{t_0} \in dx_0, \dots, X_{t_m} \in dx_m) = \nu(dx_0) \prod_{k=0}^{m-1} p_{t_{k+1}-t_k}(x_k, dx_{k+1})$$

or more formally, for $A_1, \dots, A_m \in \mathcal{B}(S)$:

$$\boxed{\mathbb{P}(X_{t_0} \in A_0, \dots, X_{t_m} \in A_m) = \int_{A_0} \nu(dx_0) \cdot \int_{A_1} p_{t_1-t_0}(x_0, dx_1) \cdots \int_{A_m} p_{t_m-t_{m-1}}(x_{m-1}, dx_m)}$$

<u>Example</u>: If $(B_t)_{t\geq 0}$ is a BM on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on $\mathbb{R}^d$, we know that $\mathrm{Law}((B_{t+s})_{t\geq 0} \mid \mathcal{F}_s) = \mathrm{Law}_{B_s}((B_t)_{t\geq 0})$ . . .

Hence the BM is a Markov process, and by its definition we can recover the transition probability:

$$p_t(x, A) = \mathbb{P}_x(B_t \in A) = \int_A \frac{e^{-\frac{1}{2t}|y-x|^2}}{(2\pi t)^{d/2}} \, dy, \quad x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d), t \geq 0$$

* <u>Markov Semigroup</u> (of Markov processes)

For a given transition probability $p : \mathbb{R}^+ \times S \times \mathcal{B}(S) \to [0,1]$ we associate the semigroup $(P_t)_{t\geq 0}$ acting on the set of measurable function $f : S \to \mathbb{R}$ (<u>bounded</u> or <u>positive</u>) so that measurable function $f: S \to \mathbb{R}$ (bounded or positive) so that

$$P_t f(x) := \int_S f(y) P_t(x, dy) = \mathbb{E}_x(f(X_t)).$$

Note: $\|P_t f\| = \sup_x |P_t f(x)| \leq \|f\| \overbrace{\int_S P_t(x,dy)}^{=1} = \|f\| \to$ P_t is a contraction

Similarly, we can consider the action of $(P_t)_{t\geq 0}$ on the set of probability measure $\nu \in \mathcal{P}(S)$ by setting, for $A \in \mathcal{B}(S)$

$$\nu P_t(A) := \int_S \nu(dx) P_t(x, A) = \mathbb{P}_\nu(X_t \in A)$$

Note. From Chapman-Kolmogorov property, $(P_t)_{t\geq 0}$ is indeed a semigroup: $P_0 = Id$ and $P_{t+s} = P_t \cdot P_s$

* Markov property (equivalent statements)

In the above setting, for $f: S \to \mathbb{R}$ measurable, bounded or positive, it holds

$$\mathbb{E}_\nu(f(X_{t+h}) | \mathcal{F}_t) = \int_S f(y) \mathbb{P}_\nu(X_{t+h} = dy | \overset{X_t}{\cancel{\mathcal{F}_t}}) \overset{\text{Markov property}}{=} \int_S f(y) P_h(X_t, dy)$$

$$= P_h f(X_t) = \mathbb{E}_{X_t}(f(X_h)), \qquad \mathbb{P}_\nu\text{-a.s.}$$

Taking the above identity (first and last) for $f = \mathbb{1}_A$, $A \in \mathcal{B}(\mathbb{R})$, we recover the Markov inequality. Thus we have an equivalent expression of the property as

$$(2) \quad \mathbb{E}_\nu(f(X_{t+h}) | \mathcal{F}_t) = \mathbb{E}_{X_t}(f(X_h)), \qquad \mathbb{P}_\nu\text{-a.s.} \quad (\text{Markov property})$$

From the Markov property ((1) or (2)), one can derive a third equivalent version of the property:

$$(3) \quad Law_\nu((X_{t+s})_{...} | \mathcal{F}_s) = Law_x((X_t)_{t\geq 0}), \quad \mathbb{P}_\nu\text{-a.s}$$

(3) $\mathcal{L}aw_\nu\left((X_{t+s})_{t\geq 0} \mid \mathcal{F}_s \right) = \mathcal{L}aw_{X_s}\left((X_t)_{t\geq 0} \right)$, P_ν - a.s

Prove as an exercise! [Suggestion: use that the law of the process is characterized by finite-dimensional distributions].

Solutions of SDE and Markov Process

- Let us consider a stochastic differential equation (SDE)

$$(*) \quad \begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \\ X_0 = x \end{cases} \qquad \text{(Cauchy problem)}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ (drift) , $\sigma : \mathbb{R}^n \to M_{n\times d}$ (diffusion coefficient) and $(B_t)_{t\geq 0}$ is a Brownian motion (BM) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with value on $\mathbb{R}^d$.

Recall: If b and σ are Lipschitz functions, namely $\exists\, L < \infty$

$$(L) \quad \begin{cases} |b(x) - b(y)| \leq L|x-y| \\ |\sigma(x) - \sigma(y)| \leq L|x-y| \end{cases} \qquad \forall\, x, y \in \mathbb{R}^n ,$$

then the SDE $(*)$ admits a unique strong solution X.
Moreover, $\forall\, T > 0$, $X \in M^2[0,T]$.

where:
- $*$ $M^2[0,T] = \{$ progressively measurable process X : $|||X|||_T < \infty \}$ and $|||X|||_T = \mathbb{E}\left[\int_0^T |X_t|^2 dt \right]$
- $*$ Strong solution : X is $(\mathcal{F}_t^B)_{t\geq 0}$-adapted
- $*$ Uniqueness is in the strongest sense: if X, X' are solutions,

* Uniqueness is in the strongest sense: if X, X' are solutions, then $\mathbb{P}(X_t = X'_t, \forall t \geq 0) = 1$ (pathwise uniqueness)

<u>Remark</u>: The mentioned result is usually stated in the more general setting where b and σ also depend on time, we the further condition: that $\exists M < \infty$

$$\begin{cases} |b(t,x)| \leq M(1+|x|) \\ |\sigma(t,x)| \leq M(1+|x|) \end{cases}$$

(sublinear growth at infinity).

However for the next result, which show that solutions of $(*)$ are homogeneous Markov processes (MP's), we have to remove the dependence on the time of b and σ.

<u>Notation</u>: Let $X^x = (X^x_t)_{t \geq 0}$ be the sol. of $(*)$, i.e.

$$X^x_t = x + \int_0^t b(X^x_s)\,ds + \int_0^t \sigma(X^x_s)\,dB_s$$

<u>Theorem</u> [SDE and MP]

In the above notation, X^x is a homogeneous MP with transition probability: $\quad P_t(x, A) := \mathbb{P}(X^x_t \in A)$

<u>Proof</u>: We have to verify the Markov property.

For $s, t \geq 0$, let's write

$$X^x_{t+s} := \underbrace{x + \int_0^s b(X^x_u)\,du + \int_0^s \sigma(X^x_u)\,dB_u}_{X^x_s} + \int_s^{t+s} b(X^x_u)\,du + \int_s^{t+s} \sigma(X^x_u)\,dB_u$$

For $y \in \mathbb{R}^n$, let $Y^y = (Y^y_t)_{t \geq 0}$ be the solution of the SDE

$$(*')\begin{cases} dY_t = b(Y_t)\,dt + \sigma(Y_t)\,d\tilde{B}_t \\ Y_0 = y \end{cases}$$

where $\tilde{B}_t := B_{t+s} - B_s$, $t \geq 0$ is a BM w.r.t. $\tilde{\mathcal{F}}_t = \mathcal{F}_{t+s}$

From the first identity we get $\boxed{X^x_{t+s} = Y_t^{X^x_s}}$. Moreover, comparing $(*)$ and $(*')$ and from the uniqueness of the solutions of SDE, X and Y have the same transition probability.

Thus, for f bounded real function, we get

since $\tilde{B}$ is indep. of $\tilde{\mathcal{F}}_0$

$$\mathbb{E}\left(f\left(X^x_{t+s}\right)\middle|\mathcal{F}_s\right) = \mathbb{E}\left(f\left(Y_t^{X^x_s}\right)\middle|\tilde{\mathcal{F}}_0\right) = \int_{\mathbb{R}^n} f(y)\cdot \mathbb{P}\left(Y_t^{X^x_s} = dy\right)$$

X and Y have same trans. prob. P

$$= \int_{\mathbb{R}^n} f(y)\, P_t\left(X^x_s, dy\right) = P_t f\left(X^x_s\right) = \mathbb{E}_{X^x_s}\left(f(X_t)\right)$$

which concludes the proof of the Markov property #

<u>Note</u>: By direct computation we can derive the Chapman-Kolmogorov identity:

$$P_{t+s}(x,A) = \mathbb{P}\left(X^x_{t+s}\in A\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_A\left(X^x_{t+s}\right)\middle|\mathcal{F}_s\right)\right)$$

$$= \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_A\left(Y_t^{X^x_s}\right)\middle|\tilde{\mathcal{F}}_0\right)\right) = \mathbb{E}\left(P_t\left(X^x_s, A\right)\right)$$

$$= \int_{\mathbb{R}^n} P_s(x, dy)\, P_t(y, A)$$

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Feller semigroup and strong Markov property

In the context of MC and t-MC we used the Markov property at random times (strong Markov property) which we were allowed

In the context of MC and [illegible] MC we used the Markov property at random times (strong Markov property), which we were allowed to use. In the continuous (uncountable) setting, this property holds only under further requirements on the semigroup $(P_t)_{t\geq 0}$.

We are going to introduce a sub-class of Markov processes, called Feller processes, which indeed satisfy the strong Markov property.

Notation: Let (S,d) a Polish space and set

* $C_0(S) := \{ f: S \to \mathbb{R} \text{, continuous and vanishing at infinity} \}$

$$= \overline{C_K}(S)$$ closure w.r.t to $\|f\| = \sup_{x\in S} |f(x)|$

where $C_K(S) = \{ f: S \to \mathbb{R} \text{, continuous with compact support} \}$

* Let $\|\cdot\|$ denote also the norm of linear operators on $C_0(S)$:

$$\text{for } A: C_0(S) \to C_0(S) \quad , \quad \|A\| = \sup_{f:\|f\|=1} \|Af\|$$

Def: A family of linear operators $(P_t)_{t\geq 0}$ is a Feller semigroup if:

1. $P_t : C_0(S) \to C_0(S)$ (invariance of the space)
2. If $f \geq 0 \implies P_t f \geq 0$, $\forall t \geq 0$ (P_t preserves positivity)
3. $\forall f \in C_0(S): \|P_t f\| \leq \|f\|$ (or $\|P_t\| \leq 1$) (P_t is a contraction)
4. $\forall t,s \geq 0$: $P_{t+s} = P_t \cdot P_s$, and $P_0 = Id$ (P_t is a semigroup)
5. $\lim_{t\downarrow 0} \|P_t f - f\| = 0$ (P_t is strongly continuous)

Remark: * If $(P_t)_{t\geq 0}$ is Markov, then 2., 3., 4. hold.

* Property 5., together 3. and 4., implies that $t \to P_t f$ is right-continuous, $\forall f \in C_0(S)$. Indeed, for h small

right-continuous, $\forall f \in C_0(S)$. Indeed, for h small

$$\|P_{t+h}f - P_t f\| \overset{4}{=} \|P_t(P_h f - f)\| \overset{3}{\leq} \|P_h f - f\| \overset{5}{<} \varepsilon(h) \to 0$$

Def: If the semigroup $(P_t)_{t\geq 0}$ of a MP $(X_t)_{t\geq 0}$ is Feller, then X is called Feller process.

Theorem [Regularity of trajectories]

If $(X_t)_{t\geq 0}$ is a Feller process, then there exists a modification $\tilde{X}$ of X (s.t. $\mathbb{P}(X_t = \tilde{X}_t) = 1 \ \forall t \geq 0$) having right-cont. trajectories, i.e $\mathbb{P}(\lim_{s \downarrow 0} \tilde{X}_{t+s} = \tilde{X}_t, \ \forall t \geq 0) = 1$

Comments:

1. Note that in general MP's do not have regular trajectories.
2. If X is Feller process, his trajectories are in $D(\mathbb{R}^+, S)$, $\mathbb{P}_\nu$-a.s.
3. It is possible to provide sufficient conditions on $(P_t)_{t\geq 0}$ s.t. the corresponding Feller process has continuous trajectories a.s.:

$$\forall \varepsilon > 0, \ \forall K \subset S \text{ compact}: \lim_{t \downarrow 0} \sup_{x \in K} \frac{1}{t} P_t(x, B_\varepsilon(x)^c) = 0$$

[Book of Revuz & Yor, Ex III 2.37]

Theorem [Strong Markov property]
Let $(X_t)_{t\geq 0}$ be a Feller process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and τ be $\mathbb{P}$-a.s. finite stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. Then, $\forall f: S \to \mathbb{R}$ measurable and bounded, and $\forall \nu \in \mathcal{P}(S)$,

$$(*) \qquad \mathbb{E}_\nu\left(f(X_{s+\tau}) \mid \mathcal{F}_\tau\right) = \mathbb{E}_{X_\tau}\left(f(X_s)\right) \qquad \mathbb{P}_\nu\text{-a.s}$$

In particular, the strong Markov property is equivalent to

$$\mathbb{P}_\nu\left(X_{s+\tau} \in A \mid \mathcal{F}_\tau\right) = \mathbb{P}_{X_\tau}\left(X_s \in A\right) \qquad \mathbb{P}_\nu\text{ - a.s}$$

$$\mathrm{Law}_\nu\left((X_{s+\tau})_{s\geq 0}\right) = \mathrm{Law}_{X_\tau}\left((X_s)_{s\geq 0}\right).$$

Proof: First, we prove it for τ taking value in a countable set, then move to the general setting considering a sequence of stopping times $(\tau_n)_{n\geq 0}$ s.t. $\tau_n \downarrow \tau$, and τ_n takes countable possible values.

1° STEP: Assume that τ takes values on $Q \subset [0, +\infty]$ countable.
To prove $(*)$, we have to show that if $A \in \mathcal{F}_\tau$, then

$$\mathbb{E}_\nu\left[f(X_{s+\tau}) \cdot \mathbb{1}_A\right] \overset{?}{=} \mathbb{E}_\nu\left(\mathbb{E}_{X_\tau}(f(X_s)) \cdot \mathbb{1}_A\right)$$

$$\mathbb{E}_\nu\left[f(X_{s+\tau}) \cdot \mathbb{1}_A\right] = \sum_{q\in Q} \mathbb{E}_\nu\left(\underbrace{\mathbb{1}_{\{\tau=q\}} \cdot \mathbb{1}_A}_{\in \mathcal{F}_q} \cdot f(X_{s+q})\right)$$

$$= \sum_{q\in Q} \mathbb{E}_\nu\left[\mathbb{1}_{\{\tau=q\}} \cdot \mathbb{1}_A\, \mathbb{E}_\nu\left(f(X_{s+q}) \mid \mathcal{F}_q\right)\right] \quad \text{Markov property}$$

$$= \sum_{q\in Q} \mathbb{E}_\nu\left[\mathbb{1}_{\{\tau=q\}} \cdot \mathbb{1}_A\, \mathbb{E}_{X_q}\left(f(X_s)\right)\right] = \mathbb{E}_\nu\left[\mathbb{1}_A \cdot \mathbb{E}_{X_\tau}\left(f(X_s)\right)\right] \qquad \#$$

2° STEP: Let τ a general stopping time and consider

0 if $\tau = 0$

STEP: let τ a general stopping time and consider

$$\tau_n := \begin{cases} 0 & \text{if } \tau = 0 \\ \frac{k+1}{2^n} & \text{if } \tau \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \end{cases}, \quad \forall n \in \mathbb{N}$$

so that τ_n take value on a countable set, $\tau_n \geq \tau$ and $\tau_n \searrow \tau$.

Then, for f bounded and continuous (hence in $C_0(S)$), and $A \in \mathcal{F}_\tau$

$$\mathbb{E}_\nu\left(f(X_{s+\tau_n}) \cdot \mathbb{1}_A\right) \overset{\text{previous step, with } A \in \mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}}{=} \mathbb{E}_\nu\left(\mathbb{E}_{X_{\tau_n}}(f(X_s)) \cdot \mathbb{1}_A\right) = \mathbb{E}_\nu\left(P_s f(X_{\tau_n}) \cdot \mathbb{1}_A\right)$$

$$\Big\downarrow n \to \infty \qquad\qquad\qquad\qquad \Big\downarrow n \to \infty$$

$$\mathbb{E}_\nu\left(f(X_{s+\tau}) \cdot \mathbb{1}_A\right) \qquad\qquad \mathbb{E}_\nu\left(P_s f(X_\tau) \cdot \mathbb{1}_A\right)$$

since $\begin{cases} \tau_n \searrow \tau \\ X \text{ is right-cont.} \\ f \text{ is cont} \end{cases}$ since $\begin{cases} \tau_n \searrow \tau \\ X \text{ is right-cont} \\ P_s f \text{ is cont. (Feller property)} \end{cases}$

Since any bounded measurable function can be approximated by bounded continuous functions, we get the complete proof. #