ADVANCED STOCHASTIC PROCESSES - 19° LECTURE

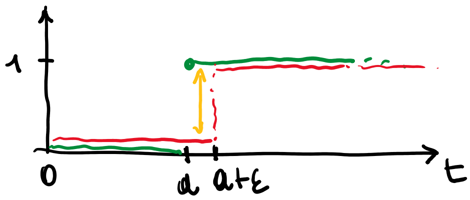
Space of trajectories

We have seen that t-MC's are defined so to have globally right-continuous trajectories $\left[X(\omega) : \underset{t}{\mathbb{R}^+} \longrightarrow \underset{X_t(\omega)}{S} \text{ is right-cont. } \forall \omega \in \Omega \right]$

The possible presence of jumps does not allow to compare two trajectories by the uniform norm, which is the most common norm for continuous trajectories.

Example: Let $x(t) := 1 - \mathbb{1}_{[0,a)}(t)$, $y(t) = 1 - \mathbb{1}_{[0,a+\varepsilon]}(t)$, $t \geq 0$

for $a, \varepsilon > 0$. Then $x, y : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are right-cont. trajectories which are very close to each other if ε is small:



However, the sup-norm $\|x-y\| := \sup_{t \in \mathbb{R}^+} |x(t) - y(t)| = 1$

hence it doesn't allow to express the closeness among x and y.

Space $D[0,T]$ [and $D(\mathbb{R}^+)$] and the Skorokhod topology

For $T \in \mathbb{R}^+$, let

$$D[0,T] := \{ x : [0,T] \longrightarrow \mathbb{R}, \underbrace{\text{right-continuous, with left limit}}_{\text{càdlàg trajectories}} \}$$

where càdlàg is the French acronym of "right-cont, left-limit".

FACT: a càdlàg trajectory has at most countably many jumps.

PLAN:

1. We first define a metric (with related topology) on $D[0,T]$.
2. We then move to define a metric on $D(\mathbb{R}^+)$

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3. We comment on the easy generalization to trajectories on a metric space (S, ρ), with (ρ: metric on S)

$$D([0,T], S) = \{x : [0,T] \to S,\ x \text{ càdlàg}\}, \quad \text{for } T \in \mathbb{R}^+$$

(and similarly $D(\mathbb{R}^+, S)$). Example: $S = \mathbb{R}^d, \mathbb{N}_0, \{0,1\}^{\mathbb{T}_L}$,

1.

Def [metric on $D[0,T]$]

Let $\Lambda = \{\lambda : [0,T] \to [0,T] : \lambda \text{ strictly increasing, continuous}, \lambda(0)=0, \lambda(T)=T\}$

For $x, y \in D[0,T]$, we define

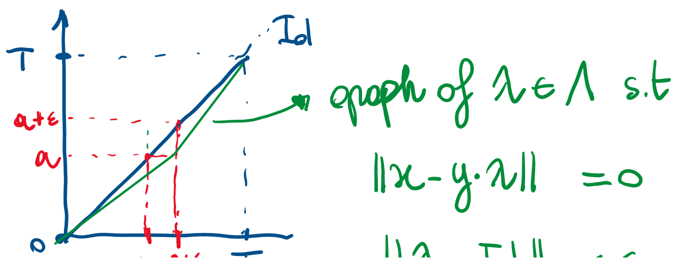
$$\boxed{d_T(x,y) := \inf_{\lambda \in \Lambda} \{\|\lambda - Id\| \vee \|x - y\lambda\|\}}$$

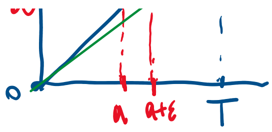
where $\|x\| := \sup_{t \in [0,T]} |x(t)|$, $\forall T \in \mathbb{R}^+$, $Id \equiv$ Identity on $[0,T]$.

__Comment__ (idea beyond the metric):

Λ is the set of possible transformations (cont. and $\nearrow$) of the time interval $[0,T]$. If in Λ there is λ close to the Id ($\|\lambda - Id\| < \varepsilon$) which allows to match the jumps among trajectories x and y ($\|x - y\lambda\| < \varepsilon'$) $\Rightarrow d_T(x,y) < \varepsilon \vee \varepsilon'$.

In the starting example, $x(t) = \mathbb{1}_{[0,a)}(t)$, $y(t) = \mathbb{1}_{[0,a+\varepsilon)}(t)$ with $t \in [0,T]$, have matching jump for λ: $\rightarrow$





$\|x - y\cdot\lambda\| = 0$

$\|\lambda - Id\| < \varepsilon$

Def: The Skorokhod topology is the topology induced by the metric d on $D[0,T]$, with open set
$$B(x_0, r) = \{x \in D[0,T] : d_T(x, x_0) < r\}, \text{ for } x_0 \in D[0,T],\ r > 0.$$

Notice that $C[0,T] \subset D[0,T] \subset \mathbb{R}^{[0,T]}$.

Some useful results

1. $D[0,T]$ is complete and separable
2. If $d_T(x^{(n)}, x) \xrightarrow[n\to\infty]{} 0$ (or $x^{(n)} \xrightarrow[n\to\infty]{}$ in $(D[0,T], d_T)$)
$$\Longrightarrow x^{n}(t) \xrightarrow[n\to\infty]{} x(t) \quad \forall\, t \text{ cont. point of } x$$

Moreover if x is cont. $\Rightarrow \|x^{(n)} - x\| \to 0$

$$\left(x^{(n)} \to x \text{ in } (C[0,T], \|\cdot\|)\right)$$

2.

Def [metric on $D(\mathbb{R}^+)$]

For $x, y \in D(\mathbb{R}^+)$, we define
$$d_\infty(x,y) := \sum_{m=1}^{\infty} \frac{1}{2^m}\, 1 \wedge d_m(\varphi_m x, \varphi_m y)$$

where $\varphi_m : D(\mathbb{R}^+) \to D[0,m]$ is s.t (left continuous in m)
$$\varphi_m x(t) = x(t)\mathbb{1}_{[0,m-1]}(t) + (m-t)\,x(t)\mathbb{1}_{(m-1,m]}(t)$$

In particular, it holds the following result (without proof)

In particular, it holds the following result (without proof)

$$\boxed{\underline{\text{Theorem}}: \quad d_\infty(x^{(n)}, x) \xrightarrow[n\to\infty]{} 0 \iff d_t(x^{(n)}, x) \xrightarrow[n\to\infty]{} 0, \quad \forall t \text{ cont. point of } x}$$

3.

Let (S, ρ) a metric space and $D([0,T], S) \subset S^{[0,T]}$ space of càdlàg functions $x : [0,T] \to S$.

To provide a metric on this space it is enough to replace the Euclidean metric $|\cdot|$ on $\mathbb{R}$ with ρ, so that

$$\boxed{d_T(x,y) = \inf_{\lambda\in\Lambda} \left\{ \|\lambda - Id\| \vee \|x - y\circ\lambda\|_\rho \right\}}, \quad \text{with}$$

$$\|x - y\circ\lambda\|_\rho = \sup_{t\in[0,T]} \rho\left(x(t), y\circ\lambda(t)\right)$$

All the mentioned properties and results hold in this general context.