ADVANCED STOCHASTIC PROCESSES - 18° LECTURE

Invariant measures and ergodic theorem

1. Invariant measures

As for MC, we are intersted in understanding the existence of invariant measures (and distributions) of a t-MC $(X_t)_{t\geq 0}$, that is, $\mu \in \mathcal{M}(S)$ s.t $\boxed{\mu P_t = \mu}$, $\forall t \geq 0$,

(where $\mu P_t(x) = \mathbb{P}_\mu(X_t = x)$ if $\mu \in \mathcal{P}(S)$)

From the Kolmogorov forward equation:

$$\frac{d}{dt}(\mu P_t) = \mu \frac{d}{dt} P_t = \mu P_t \cdot L = \mu L \text{ (if } \mu \text{ is stationary)}$$

As a consequence μ is stationary $\iff$ $\boxed{\mu L = 0}$

Explicitly: $\sum_{x \in S} \mu(x) L(x,y) = 0$

$$\iff \boxed{\mu(y) \cdot \overset{\text{arrows exiting from } y}{\lambda_y} = \sum_{x \neq y} \mu(x) \overset{\text{arrows entering in } y}{q(x,y)}} \;\; *$$

Example:

Consider a birth and death process on $\mathbb{N}_0$

$$\begin{tikzcd} 0 \arrow[r, bend left, "b_0"] & 1 \arrow[l, bend left, "d_1"] \arrow[r, bend left, "b_1"] & 2 \arrow[l, bend left, "d_2"] \arrow[r, bend left, "b_2"] & 3 \arrow[l, bend left, "d_3"] \arrow[r, bend left, "b_3"] & 4 \arrow[l, bend left, "d_4"] & \cdots \end{tikzcd}$$

Setting $\mu L = 0$, from $*$ we get:

1. $\mu(0) b_0 = \mu(1) d_1$

2. $\forall k > 0$: $\mu(k)(b_k + d_k) = \mu(k-1) b_{k-1} + \mu(k+1) d_{k+1}$

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By recursively subtracting two consecutive equation $((k+1)^{th}$ eq $-(k)^{th}$ eq.$)$, and using 1., we get $\mu(k)b_k = \mu(k+1)d_{k+1} \iff \mu(k+1) = \frac{b_k}{d_{k+1}}\mu(k)$

$$\Rightarrow \mu(k) = \left(\prod_{j=1}^{k} \frac{b_j}{d_{j+1}}\right)\mu(0) \quad , \quad \forall k>0$$

If $c = \sum_{k\geq 1}\left(\prod_{j=1}^{k}\frac{b_j}{d_{j+1}}\right) < \infty$ then μ is a finite measure, unique up to a multiplicative constant (choice of $\mu(0)$)

Then μ can be normalized to a distribution π s.t.

$\pi(0) = \frac{1}{c+1}$.

$\boxed{2.}$ Recurrence, transience and existence of an invariant measure

Def: Let $(X_t)_{t\geq 0}$ be a t-MC on S, with embedded MC $(\tilde{X}_n)_{n\geq 0}$.

- $(X_t)_{t\geq 0}$ is irreducible $\iff (\tilde{X}_n)_{n\geq 0}$ is irreducible
- $(X_t)_{t\geq 0}$ is recurrent (transient) $\iff (\tilde{X}_n)_{n\geq 0}$ is recurrent (transient)
- A state $x\in S$ is t-positive recurrent if $\mathbb{E}_x(R_x) < \infty$,

  $R_x = \inf\{t>0 : X_t = x \ , \ \underbrace{\exists s<t \text{ with } X_s \neq x}_{\hookrightarrow \text{ otherwise } \mathbb{E}_x(R_x)=0}\}$ = Return time to x

  Otherwise is null recurrent.

Note: positive recurrence is not equivalent among X and $\tilde{X}$.

$\boxed{3.}$ Main Results

Notation: let $L_t(x) = \int_0^t \mathbb{1}_{\{X_s = x\}}\, ds$ (local time in x up to t)

(corresponding to $N_n(x)$ in MC)

(corresponding to $N_m(x)$ in MC)

Theorem [Invariant measure of t-MC]

If $(X_t)_{t\geq 0}$ is a irreducible and recurrent t-MC on S, then $\forall x \in S$

$$\nu_x(y) := \mathbb{E}_x\left(L_{R_x}(y)\right) = \mathbb{E}_x\left(\int_0^{R_x} \mathbb{1}_{\{X_s = y\}}\, ds\right)$$

is an invariant measure for X (unique up to multiplicative factors), and satisfies: $\nu_x(y) > 0 \ \forall y \in S$, and

(*) $\boxed{\nu_x(y) = \dfrac{\mu_x(y)}{\lambda_y}}$, where $\mu_x(y) = \mathbb{E}_x\left(N_{\tilde{\tau}_x}(y)\right)$

embedded MC

invariant measure for $\tilde{X}$, $\tilde{\tau}_x = \inf\{m > 0 : \tilde{X}_m = x\}$

Consequences and comments

1. The measure ν_x is not necessarily finite (as well as μ_x). Indeed
$$\sum_{y \in S} \nu_x(y) = \mathbb{E}_x\left(\sum_{y \in S} L_{R_x}(y)\right) = \mathbb{E}_x(R_x)$$
(may be ∞ under solely recurrence)

2. If x is t-positive recurrent $\Rightarrow \pi_x(y) = \dfrac{\nu_x(y)}{\mathbb{E}_x(R_x)}$, $y \in S$, is $\mathcal{P}(S)$

and it is s.t. $\boxed{\pi_x(x) = \dfrac{1}{\lambda_x \cdot \mathbb{E}_x(R_x)}}$ unique invariant distribution

since $\nu_x(x) = \mathbb{E}_x\left[\int_0^{R_x} \mathbb{1}_{\{X_s = x\}}\, ds\right] = \mathbb{E}_x\left[\int_0^{\tau_1} ds\right] = \mathbb{E}_x(\tau_1) = \dfrac{1}{\lambda_x}$

In that case, from uniqueness up to multiplicative constant of ν_x, we get: $\forall x, y \in S: \pi_x = \pi_y = \pi$, and hence the unique invariant distribution of $(X_t)_{t\geq 0}$ is $\pi(x) = \dfrac{1}{\lambda_x \mathbb{E}_x(R_x)}$.

3. From (*) $\nu_x(y)\lambda_y = \mu_x(y)$. Then:

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$$\begin{bmatrix} (X_t)_{t\geq 0} \text{ t-positive recurrent} \\ \text{if } \sum_{y\in S} \nu_x(y) < \infty \end{bmatrix} \underset{\text{not comparable in general}}{\not\longleftrightarrow} \begin{bmatrix} (\tilde{X}_n)_{n\geq 0} \text{ positive recurrent} \\ \text{if } \sum_{y\in S} \underbrace{\mu_x(y)}_{=\nu_x(y)\cdot\lambda_y} < \infty \end{bmatrix}$$

<u>Proof</u> (of Thm [inv. measure for t-MC])

- $\nu\in\mathcal{M}(S)$ is invariant measure for $L \iff \lambda_x \nu(x) = \sum_{y\neq x}\nu(y)q(y,x)\cdot\frac{\lambda_y}{\lambda_y}$

$\iff \lambda_x\nu(x) = \sum_{y\neq x}\lambda_y\nu(y)\,\tilde{P}_{y,x} \iff (\lambda_y\nu(y))_{y\in S}$ is invariant for $\tilde{P}$.

Since $\tilde{P}$ is recurrent and irreducible (by hypothesis), then the only invariant measures for $\tilde{P}$ (unique up to mult. constants) are

$$\mu_x(y) = \mathbb{E}_x\left(N_{\tilde{\tau}_x}(y)\right), \quad y\in S.$$

Then: $\begin{cases} * \ \nu_x(y) = \dfrac{\mu_x(y)}{\lambda_y} \text{ are the only invariant measures (up to multiplicative constant)} \\ * \ \nu_x(y) > 0 \ \forall y\in S \ (\text{since } \mu_x(y)>0,\ \lambda_y>0) \end{cases}$

- Some extra work is needed to prove that

$$\nu_x(y) = \mathbb{E}_x\left(L_{R_x}(y)\right)$$

" Idea: $\mathbb{E}_x\left(L_{R_x}(y)\right) = \mathbb{E}_x\left(N_{\tilde{\tau}_x}(y)\right)\cdot\mathbb{E}(T_y) = \dfrac{\mu_x(y)}{\lambda_y}$ "

$$\mathbb{E}_x\left(L_{R_x}(y)\right) = \mathbb{E}_x\left(\int_0^{R_x} \mathbb{1}_{\{X_s=y\}}\,ds\right) = \mathbb{E}_x\left(\sum_{k=0}^{N_{\tilde{\tau}_x}(y)} T_y^{(k)}\right)$$

$T_y^{(k)}$ = holding times in y at the k-th visit $\Rightarrow$ all indep. and with same law $Exp(\lambda_y)$

Formal proof on Bremaud "Markov Chains" - Chap. 8, Thm 5.1. #

<u>Theorem</u> [ergodic for t-MC]

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If $(X_t)_{t\geq 0}$ is an irreducible and positive recurrent t-MC and π its unique invariant distribution. Then, $\forall \mu \in \mathcal{P}(S)$:

$\| \mu P_t - \pi \|_{TV} \xrightarrow[t\to\infty]{} 0$ (equiv: $\sum_{y\in S} |P_t(x,y) - \pi(y)| \xrightarrow[t\to\infty]{} 0$, $\forall x \in S$)

Proof: Consider the M.C. $Y_n := X_n$, $n \in \mathbb{N}$ (skeleton M.C.)
↳ positions of $(X_t)_{t\geq 0}$ only for discrete time

- By construction, and from the hypotheses over $(X_t)_{t\geq 0}$, $(Y_n)_{n\in\mathbb{N}}$ is a MC. irreducible, positive recurrent and aperiodic, since the holding time in a state is exponential distributed and hence Y has self-jumps to each state.
- Then Y satisfies an ergodic theorem, implying that in a finite time two MC with different initial distributions will couple a.s..
- The two corresponding t-MC (having different initial distributions) will also couple a.s. in a finite time, as their skeletons do. #