

Invariant measures and ergodic theorem

1. Invariant measures

As for MC, we are interested in understanding the existence of invariant measures (and distributions) of a t-MC $(X_t)_{t \geq 0}$,

that is, $\mu \in \mathcal{M}(S)$ s.t. $\mu P_t = \mu, \forall t \geq 0$,

(where $\mu P_t(x) = \mathbb{P}_\mu(X_t = x)$ if $\mu \in \mathcal{P}(S)$)

From the Kolmogorov forward equation:

$$\frac{d}{dt}(\mu P_t) = \mu \frac{d}{dt} P_t = \mu P_t \cdot L = \mu L \quad (\text{if } \mu \text{ is stationary})$$

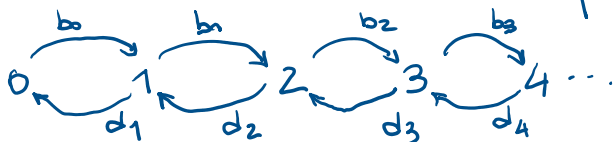
As a consequence μ is stationary $\iff \mu L = 0$

Explicitly: $\sum_{x \in S} \mu(x) L(x, y) = 0$

$$\iff \mu(y) \cdot \overset{\substack{\text{arrows exiting} \\ \text{from } y}}{r_y} = \sum_{x \neq y} \overset{\substack{\text{arrows entering} \\ \text{in } y}}{\mu(x) q(x, y)} \quad *$$

Example:

Consider a birth and death process on \mathbb{N}_0



Setting $\mu L = 0$, from * we get:

$$1. \mu(0) b_0 = \mu(1) d_1$$

$$2. \forall k \geq 0: \mu(k) (b_k + d_k) = \mu(k-1) b_{k-1} + \mu(k+1) d_{k+1}$$

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By recursively subtracting two consecutive equations ($(k+1)^{\text{th}}$ eq. - k^{th} eq.),

and using 1, we get $\mu(k) b_k = \mu(k+1) d_{k+1} \Leftrightarrow \mu(k+1) = \frac{b_k}{d_{k+1}} \mu(k)$

$$\Rightarrow \mu(k) = \left(\prod_{j=1}^k \frac{b_j}{d_{j+1}} \right) \mu(0), \quad \forall k \geq 0$$

If $c = \sum_{k \geq 0} \left(\prod_{j=1}^k \frac{b_j}{d_{j+1}} \right) < \infty$ then μ is a finite measure, unique up to a multiplicative constant (choice of $\mu(0)$)

Then μ can be normalized to a distribution π s.t.

$$\pi(0) = \frac{1}{c+1} -$$

2. Recurrence, transience and existence of an invariant measure

Def: let $(X_t)_{t \geq 0}$ be a t-MC on S , with embedded MC $(\tilde{X}_n)_{n \geq 0}$.

- $(X_t)_{t \geq 0}$ is recducible $\Leftrightarrow (\tilde{X}_n)_{n \geq 0}$ is recducible

- $(X_t)_{t \geq 0}$ is recurrent $\Leftrightarrow (\tilde{X}_n)_{n \geq 0}$ is recurrent
(transient) (transient)

- A state $x \in S$ is t-positive recurrent if $\mathbb{E}_x(R_x) < \infty$,

$$R_x = \inf \{ t > 0 : X_t = x, \exists s < t \text{ with } X_s \neq x \} = \text{Return time to } x$$

\hookrightarrow otherwise $\mathbb{E}_x(R_x) = \infty$

Otherwise is null recurrent.

Note: positive recurrence is not equivalent among X and \tilde{X} .

3. Main Results

Notation: let $L_t(x) = \int_0^t \mathbb{1}_{\{X_s = x\}} ds$ (local time in x up to t)
(corresponding to $N_n(x)$ in MC)

(corresponding to $N_m(x)$ in MC)

Theorem [Invariant measure of t-MC]

If $(X_t)_{t \geq 0}$ is a irreducible and recurrent t-MC on S , then $\forall x \in S$

$$\nu_x(y) := \mathbb{E}_x \left(L_{R_x}(y) \right) = \mathbb{E}_x \left(\int_0^{R_x} \mathbb{1}_{\{X_s=y\}} ds \right)$$

is an invariant measure for X (unique up to multiplicative factors),

and satisfies: $\nu_x(y) > 0 \forall y \in S$, and

(*) $\boxed{\nu_x(y) = \frac{\mu_x(y)}{\lambda_y}}$, where $\mu_x(y) = \mathbb{E}_x(N_{\tilde{C}_x}(y))$

\downarrow
invariant measure for \tilde{X} , \leftarrow embedded MC
 $\tilde{C}_x = \inf \{n > 0 : \tilde{X}_n = x\}$

Consequences and comments

1. The measure ν_x is not necessarily finite (as well as μ_x). Indeed

$$\sum_{y \in S} \nu_x(y) = \mathbb{E}_x \left(\sum_{y \in S} L_{R_x}(y) \right) = \mathbb{E}_x(R_x) \quad \left(\begin{array}{l} \text{may be } \infty \text{ under} \\ \text{recurrent} \end{array} \right)$$

2. If x is t-positive recurrent $\Rightarrow \pi_x(y) = \frac{\nu_x(y)}{\mathbb{E}_x(R_x)}$, $y \in S$, is $\mathcal{P}(S)$

and it is st. $\boxed{\pi_x(x) = \frac{1}{\lambda_x \cdot \mathbb{E}_x(R_x)}}$ unique invariant distribution

since $\nu_x(x) = \mathbb{E}_x \left[\int_0^{R_x} \mathbb{1}_{\{X_s=x\}} ds \right] = \mathbb{E}_x \left[\int_0^{\tilde{C}_1} ds \right] = \mathbb{E}_x(\tilde{C}_1) = \frac{1}{\lambda_x}$

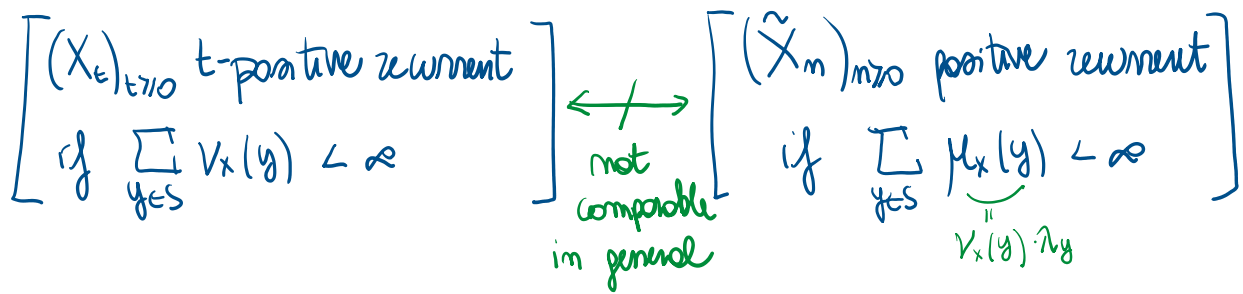
In that case, from uniqueness up to multiplicative constant of ν_x ,

we get: $\forall x, y \in S: \pi_x = \pi_y = \pi$, and hence the unique

invariant distribution of $(X_t)_{t \geq 0}$ is $\pi(x) = \frac{1}{\lambda_x \mathbb{E}_x(R_x)}$.

3. From (*) $\nu_x(y) \lambda_y = \mu_x(y)$. Then:

3. From (*) $v_x(y) \lambda_y = \mu_x(y)$. Then:



Proof (of Thm [inv. measure for t-MC])

• $\nu \in \mathcal{M}(S)$ is invariant measure for $L \iff \lambda_x \nu(x) = \sum_{y \neq x} \nu(y) q(y, x) \frac{\lambda_y}{\lambda_x}$

$\iff \lambda_x \nu(x) = \sum_{y \neq x} \lambda_y \nu(y) \tilde{P}_{y,x} \iff (\lambda_y \nu(y))_{y \in S}$ is invariant for \tilde{P} .

Since \tilde{P} is recurrent and irreducible (by hypothesis), then the only invariant measures for \tilde{P} (unique up to mult. constants) are

$$\mu_x(y) = \mathbb{E}_x(N_{\tilde{E}_x}(y)), \quad y \in S.$$

Then: $\left\{ \begin{array}{l} * v_x(y) = \frac{\mu_x(y)}{\lambda_y} \text{ are the only invariant measures (up to multiplicative constant)} \\ * v_x(y) > 0 \quad \forall y \in S \text{ (since } \mu_x(y) > 0, \lambda_y > 0) \end{array} \right.$

• Some extra work is needed to prove that

$$v_x(y) = \mathbb{E}_x(L_{R_x}(y))$$

" Idea: $\mathbb{E}_x(L_{R_x}(y)) = \mathbb{E}_x(N_{\tilde{E}_x}(y)) \cdot \mathbb{E}(T_y) = \frac{\mu_x(y)}{\lambda_y}$ "

$\mathbb{E}_x\left(\int_0^{R_x} \mathbb{1}_{\{X_s=y\}} ds\right) = \mathbb{E}_x\left(\sum_{k=0}^{N_{\tilde{E}_x}(y)} T_y^{(k)}\right)$ " holding times in y at the k -th visit \Rightarrow all indep. and with same law $\text{Exp}(\lambda_y)$ "

Formal proof on Bremaud "Markov Chains" - Chap. 8, Thm 5.1. \neq

Theorem [ergodic for t-MC]

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If $(X_t)_{t \geq 0}$ is an irreducible and positive recurrent t-MC and π its unique invariant distribution. Then, $\forall \mu \in \mathcal{P}(S)$:

$$\| \mu P_t - \pi \|_{TV} \xrightarrow{t \rightarrow \infty} 0 \quad (\text{equiv: } \sum_{y \in S} |P_t(x, y) - \pi(y)| \xrightarrow{t \rightarrow \infty} 0, \forall x \in S)$$

Proof: Consider the M.C. $Y_m := X_m$, $m \in \mathbb{N}$ (skeleton M.C.).

- By construction, and from the hypotheses over $(X_t)_{t \geq 0}$, \hookrightarrow portions of $(X_t)_{t \geq 0}$ only for discrete time $(Y_m)_{m \in \mathbb{N}}$ is a M.C. irreducible, positive recurrent and aperiodic, since the holding time in a state is exponential distributed and hence Y has self-jumps to each state.
 - Then Y satisfies an ergodic theorem, implying that in a finite time two MC with different initial distributions will couple a.s..
 - The two corresponding t-MC (having different initial distributions) will also couple a.s. in a finite time, as their skeletons do. \neq
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