ADVANCED STOCHASTIC PROCESSES - 17' LECTURE

Recall: under proper assumptions ($\lambda = \sup_{x \in S} \lambda_x < \infty$):

1. A t-MC is uniquely determined by its generator L, i.e.

$$P_t = e^{tL}$$

2. Following the proof, a t-MC can be constructed as a uniform MC built over

   $\hat{P}_{x,y} = \frac{q_{xy}}{\lambda}$, if $x \neq y$

   - a subordinated MC $(\hat{X}_n)_{n \in \mathbb{N}}$
   - a Poisson clock $(N_t)_{t \geq 0}$

   independent

   $PP(\lambda)$

There is an equivalent representation of (X_t) which is more convenient in application because it allows to couple $\hat{X}$ and $\hat{N}$ in a unique object and holds even when $\lambda = +\infty$.

Transition times and the embedded MC

For a t-MC $(X_t)_{t \geq 0}$, consider the transition times $(T_n)_{n \geq 0}$ s.t

$$T_0 = 0 \quad , \quad T_n = \inf\{t > T_{n-1} : X_t \neq X_{T_{n-1}}\} \quad \forall n \geq 1$$

so that $X_t = X_{T_{n-1}} \quad \forall t \in [T_{n-1}, T_n)$

Remark: from assumptions A (non-explosion), $T_n \xrightarrow[n\to\infty]{} +\infty$, i.e there is not a "last jump" in a finite time.

Given a t-MC $(X_t)_{t \geq 0}$, let $\tilde{X}_n = X_{T_n}$, $\forall n \geq 0$.

Theorem [embedded MC]:

In the above setting, it holds:

a. $(\tilde{X}_n)_{n\geq 0}$ is a MC on S, called embedded MC, with

trans. prob: $\tilde{P}_{x,y} = \dfrac{q(x,y)}{\lambda_x}$, $\forall y \neq x$ (and 0 if $y=x$)

b. Given $(\tilde{X}_n)_{n\geq 0}$, the transition time increments $(T_{n+1}-T_n)_{n\geq 0}$ are all independent and s.t.

$$\mathcal{L}aw\left(T_{n+1}-T_n \mid \tilde{X}_n = x\right) = \mathrm{Exp}(\lambda_x) \quad , \forall x \in S$$

Proof (ideas):

a. $(\tilde{X}_n)$ can be constructed from the subordinated MC $(\hat{X}_n)$, by neglecting the self-jump to a state (so that $\tilde{P}_{x,x}=0$ $\forall x \in S$).
Hence it inherits the Markov property from $(\hat{X}_n)$ (verify!), and for $x \neq y$ we get

$$\tilde{P}_{x,y} = \mathbb{P}(\tilde{X}_n = y \mid \tilde{X}_{n-1} = x) = \sum_{m=0}^{\infty} (\hat{P}_{x,x})^m \cdot \hat{P}_{x,y} = \frac{\hat{P}_{x,y}}{1-\hat{P}_{x,x}}$$

$$= \frac{q(x,y)}{\lambda_x} \quad \leftarrow \left(\hat{P}_{x,y} = \frac{q(x,y)}{\lambda} \, , \, \hat{P}_{x,x} = 1 - \frac{\lambda_x}{\lambda} \right)$$

#

b. We first consider T_1, and assume that $X_0 = \tilde{X}_0 = x$. Then, $\forall t>0$

$$\mathbb{P}_x(T_1 > t) = \mathbb{P}_x\Big(\underbrace{\bigcup_{k=0}^{\infty}}_{\text{disjoint}} \{ N_t = k \, , \, \hat{X}_j = x \ \forall j = 1,..,k \} \Big)$$

indep.

(N_t) is $PP(\lambda)$, $(\hat{X}_n)$ subordinated MC

$$= \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \left(1-\frac{\lambda_x}{\lambda}\right)^k = e^{-\lambda t} e^{(\lambda-\lambda_x)t} = e^{-\lambda_x t}$$

$$\Longrightarrow \mathcal{L}aw(T_1 \mid \tilde{X}_0 = x) = \mathrm{Exp}(\lambda_x)$$

Using the strong Markov property of $(\tilde{X}_n)$ and independence of the

Using the strong Markov property of $(\tilde{X}_n)$ and independence of the inter-arrivals of (N_t), one can prove similarly that

$$\mathrm{Law}\left(\tau_{n+1}-\tau_n \mid \tilde{X}_n = x\right) = \mathrm{Exp}(\lambda_x)$$

and the independence of $(\tau_{n+1}-\tau_n)_{n\geq 0}$, given $(\tilde{X}_n)$. #

Graphical construction of t-MC

The previous theorem suggests the following graphical construction, which is the most common in application:

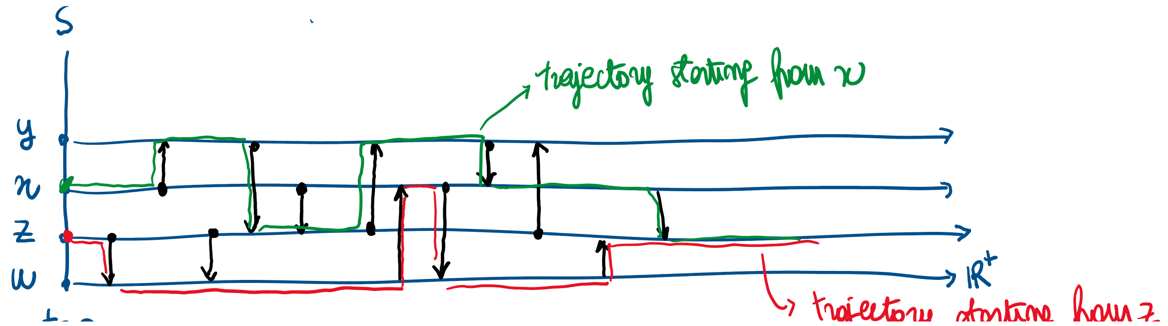
1. $\forall x \in S$, let us consider a $PP(\lambda_x)$, denoted $N^x = (N_t^x)_{t\geq 0}$, so that N^x, $x \in S$ are all independent.

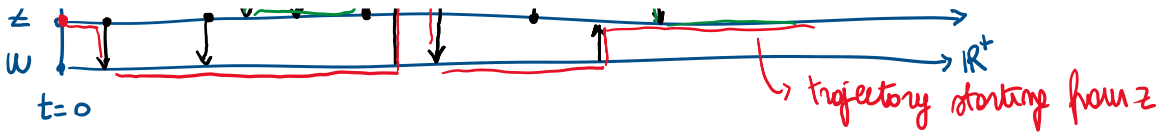
2. Each arrival of N^x corresponds to a jump in a rate (mark of the arrival) $y \neq x \in S$ with probability $\tilde{P}_{x,y} = \frac{q(x,y)}{\lambda_x}$ $\Longrightarrow$ the process N^x can be seen as a marked PP, and decomposed as $N_t^x = \sum_{y\neq x} N_t^{x,y}$,

   where $N_t^{x,y} = \#$ of transition $x \rightsquigarrow y$ in $[0,t]$ is $PP(q(x,y))$.

Graphically, for each state $x \in S$, we sample N^x, and draw an arrow among x and y at each arrival of $N^{x,y}$ $(\forall y \in S)$.

Then we get a trajectory starting from a state (say x) at $t=0$, and following the time line and the arrows as in the picture:





Remark: this construction holds also thanks to the absence of memory of the exponential law, so that we can initiate all the PP's $(N_x)_{x\in S}$ at $t=0$, instead of resetting the time after every transition.

Conclusion: $\begin{cases} q_{x,y} = \text{rate of the Exponential time to jump from } x \text{ to } y \\ \qquad\qquad (\text{jump rate}) \\ \lambda_x = \text{rate of the Exponential holding time in } x \\ \qquad\qquad (\text{holding rate}) \end{cases}$

[Indeed, if $T_{x,y} \sim Exp(q_{x,y})$, $\forall\, y \neq x \in S$, independent, then $T_x = \min_{y\neq x} \{T_{x,y}\} \sim Exp\left(\sum_{y\neq x} q_{x,y}\right) = Exp(\lambda_x)$ is precisely the time spent in x before jumping to a new state.]

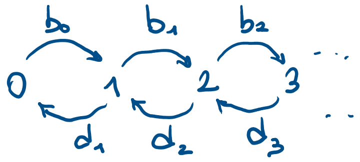
Graphical construction of t-MC: Examples

Example 1: Birth and death processes

Consider a t-MC $(X_t)_{t\geq 0}$ on $\mathbb{N}_0$ with jump rates

$$q(x,y) = \begin{cases} b_x & \text{if } y = x+1 \\ d_x & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } \underset{\substack{\uparrow \\ \text{birth rate} \\ \text{at } x}}{b_x}, \underset{\substack{\uparrow \\ \text{death rate} \\ \text{at } x}}{d_x} > 0$$

which can be sketched:



Then:

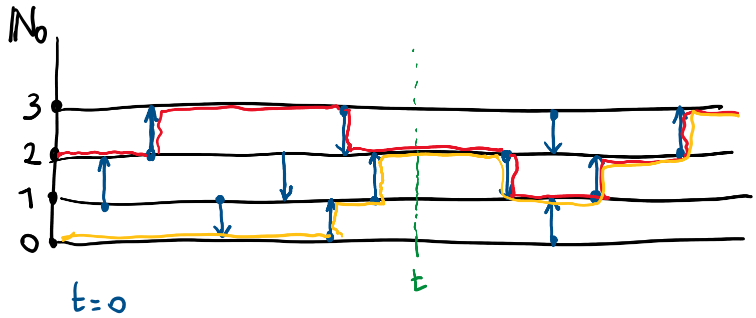
$\lambda_x = b_x + d_x \quad \forall x \in \mathbb{N}_0 \quad (\text{with } d_0 = 0)$

The embedded MC $(\tilde{X}_n)$ has trans prob $\tilde{P}_{x,y} = \begin{cases} \frac{b_x}{\lambda_x} & \text{if } y = x+1 \\ \frac{d_x}{\lambda_x} & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}$

* Graphical construction

$\mathbb{N}_0$,

* Graphical construction $\qquad\qquad\qquad\qquad \big(0 \quad$ otherwise



trajectories starting from 0, and 2

<u>Interpretation</u>: They model the evolution of the size of a population (hence the name) but they are also used to describe queuing system, where "birth" = "arrival of a client", "death" = "end service of a client".

<u>Special cases</u>:

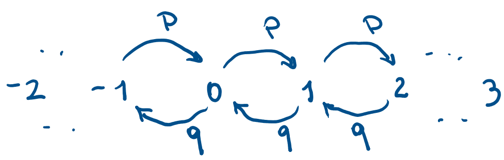
* if $b_x = b$ and $d_x = 0$, $\forall x \in \mathbb{N}_0 \Rightarrow PP(b)$
* if $d_x = 0 \ \forall x \in \mathbb{N}_0 \Rightarrow$ pure birth process = counting process with $Exp(b_x)$ arrival times
* if $b_x = b$, and $d_x = d$, $\forall x \in \mathbb{N}_0 \Rightarrow$ queue system $M/M/1$
  (1 server; Markovian arrivals at rate b; Markovian service at rate d)

Example 2. <u>Continuous time RW on $\mathbb{Z}$</u> $\longrightarrow$ or on a graph $G=(V,E)$

Consider a t-MC $(X_t)_{t\geq 0}$ on $\mathbb{Z}$ with jump rates

$$q(x,y) = \begin{cases} p & \text{if } y = x+1 \\ q & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } p, q \geq 0$$

which can be sketched:

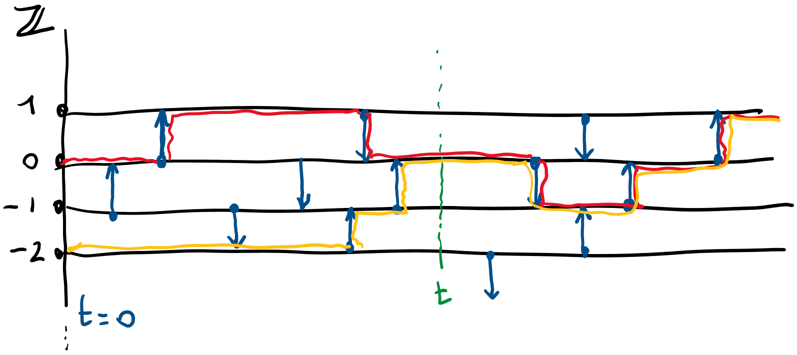


Then:

- $\lambda = \lambda_x = p + q \quad \forall x \in \mathbb{Z}$
- The embedded MC $(\tilde{X}_n)$ has trans prob $\tilde{P}_{xy} = \begin{cases} \frac{p}{p+q} & \text{if } y = x+1 \\ \frac{q}{p+q} & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}$
- Graphical construction

$p+q$ if ... / 0 otherwise

- Graphical construction



At each $x \in \mathbb{Z}$, there are indep. PP(p) and PP(q): at each arrival we insert a directed arrow

With this construction, we can also consider the motion of many random walkers, and let them interact, as in the next example.

Example 3: Interacting particle systems on G

- Let $S = \{0,1\}^V$ = configuration space, $G = (V, E)$ finite graph with element $\eta = (\eta(x))_{x \in V}$ s.t. $\eta(x) \in \{0,1\}$

$\rightarrow \begin{cases} \eta(x) = 1 & \text{if the site } x \text{ is occupied / infected / favorable} \\ \eta(x) = 0 & \text{if the site } x \text{ is vacant / healthy / against} \end{cases}$

exclusion process ← "interacting walkers" | contact process | voter model

- We consider dynamics that each step change the value of η only in 1 or 2 points:

Notation: For $\eta \in S$ and $x \in V$, let η^x s.t. $\eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x \\ 1-\eta(x) & \text{if } z = x \end{cases}$ = flip at x

$\eta = \dots 0101\underset{x}{1}\dots \longrightarrow \eta^x = \dots 0100\underset{x}{1}\dots$

For $\eta \in S$ and $x, y \in V$, let η^{xy} s.t. $\eta^{xy}(z) = \begin{cases} \eta(z) & \text{if } z \neq x,y \\ \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y \end{cases}$

$\eta = \dots 01\underset{x}{0}\underset{y}{1}1\dots \longrightarrow \eta^{xy} = \dots 01\underset{x}{1}\underset{y}{0}1$

$\boxed{A.}$ Exclusion process: is a t-MC on S with transition rates

$q(\eta, \eta') = \begin{cases} 0 & \text{if } \eta' \neq \eta^{xy} \text{ for all } x,y \in V \end{cases}$

[A] Exclusion process: is a t-MC on S with transition rates

$$q(\eta,\eta') = \begin{cases} 0 & \text{if } \eta' \neq \eta^{x,y}, \text{ for all } x,y \in V \\ \lambda_{x,y}\,\eta(x)(1-\eta(y)) & \text{if } \eta' = \eta^{x,y} \end{cases}$$

where $(\lambda_{x,y})_{x,y\in G}$ are transition rates of a cont. time RW on V

↳ For example $\lambda_{x,y} = \begin{cases} 1 & \text{if } y \sim x \text{ (neighbors in } G) \\ 0 & \text{otherwise} \end{cases}$ → simple symmetric RW on G

↳ $q(\eta,\eta^{x,y}) > 0 \iff \eta(x)=1,\ \eta(y)=0$ and $\lambda_{x,y} > 0$

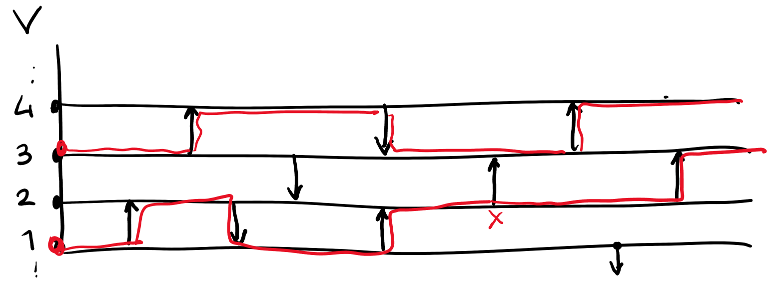
and it provides the transition $\underset{x\ y}{1\ 0} \xrightarrow{\lambda_{xy}} \underset{x\ y}{0\ 1}$ among site x (where a particle is placed) to a reachble ($\lambda_{x,y}>0$) site y, which is free.

This model describes the motion of many walkers (particles), each avoiding each other (exclusion interaction). It is a basic model for transport phenomena, such as traffic flow.

* Generator:

$$L f(\eta) = \sum_{\eta'} \underbrace{q(\eta,\eta')}_{0 \text{ for most of } \eta'} \left[f(\eta') - f(\eta) \right] = \sum_{x,y\in\mathbb{T}_L} \lambda_{x,y}\,\eta(x)(1-\eta(y)) \left[f(\eta^{x,y}) - f(\eta) \right]$$

* Graphical construction (for $\lambda_{xy} = \begin{cases} 1 & \text{if } y \sim x \\ 0 & \text{otherwise} \end{cases}$)



At each $x \in V$, there are indep. PP $(\lambda_{x,y})$: at each arrival we draw a directed arrow.
Constraint: we can not move in already occupied path

[B] Contact process: is a t-MC on S with trans. rates

$$q(\eta,\eta') = \begin{cases} 0 & \text{if } \eta' \neq \eta^{x},\ \forall x \in V \\ 1 & \text{if } \eta' = \eta^{x} \text{ and } \eta(x) = 1 \\ \lambda \sum_{y\sim x} \eta(y) & \text{if } \eta' = \eta^{x} \text{ and } \eta(x) = 0 \end{cases}$$

$$\lambda \sum_{y\sim x} \eta(y) \qquad \text{if } \eta' = \eta^x \text{ and } \eta(x) = 0$$

where $\lambda > 0$

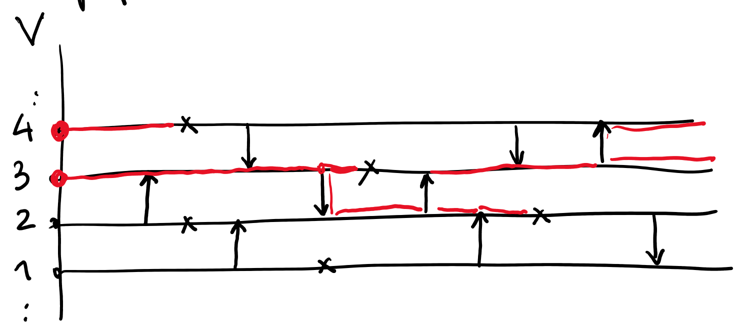
$\hookrightarrow \underset{x}{1} \xrightarrow{\lambda} \underset{x}{0}$ (healing) , $\underset{x}{0} \xrightarrow{\lambda \sum \eta(y)} \underset{y}{1}$ (infection)

- Notice that the infection rates is proportional to # of infected neighbors.
- This dynamics has absorbing state $\underline{0}$ (all 0's configuration).

* Generator:

$$Lf(\eta) = \sum_{x\in V} \left[\eta(x) + \lambda(1-\eta(x)) \sum_{y\sim x} \eta(y)\right]\left(f(\eta^x) - f(\eta)\right)$$

* Graphical construction



At each point $x \in V$:

- $PP(1) \to$ healing events marked with crosses
- $PP(\lambda) \to$ infection event of neighbor y $\forall y \sim x$