ADVANCED STOCHASTIC PROCESSES - 16° LECTURE

We want to show that under quite general assumptions, any t-MC can be realised as a uniform MC, and then identified by an operator (the generator) which allows to reconstruct the Markov semigroup $(P_t)_{t\geq 0}$.

Transition rates and generator

Def [transition rates]: $\forall x \neq y \in S$, let

* $q(x,y) := \lim_{t\downarrow 0} \frac{1}{t} \mathbb{P}_x(X_t = y) = \lim_{t\downarrow 0} \frac{1}{t} P_t(x,y) \geq 0$ (transition rate among x and y)

(which is well defined when the limit exists)

* $\lambda_x := \sum_{y\neq x} q(x,y)$ (transition rate from x)

Assumptions (A)
* $q(x,y)$ exists $\forall x\neq y$
* $\lambda_x < \infty$ $\forall x \in S$
* $\sup_{x\in S} \lambda_x < \infty$

condition for non-explosion, i.e. simultaneous or infinite jumps

Def [generator]: We complete $(q(x,y))_{x\neq y \in S}$ to a matrix L that has sum 0 on each line, hence setting

* $L(x,y) = q(x,y)$, $\forall x\neq y$
* $L(x,x) = -\lambda_x = -\sum_{y\neq x} q(x,y)$

L is called generator of the t-MC with semigroup $(P_t)_{t\geq 0}$ and it is shortly given by: $L = \lim_{t\downarrow 0} \frac{1}{t}(P_t - \underbrace{Id}_{=P_0})$ (*)

The generator L acts on real functions on S, $L: \mathbb{R}^S \to \mathbb{R}^S$, $f \to Lf$, as

The generator L acts on real functions on S, $L: \mathbb{R}^S \to \mathbb{R}^S$, $f \to Lf$, as

$$Lf(x) = \sum_{y \in S} L(x,y) f(y) = \sum_{y \in S} q(x,y) \big(f(y) - f(x)\big)$$

Examples:

- If $(N_t)_{t \geq 0}$ is a $PP(\lambda)$, then $L(x,y) = \lambda\big(\mathbb{1}_{\{x+1\}}(y) - \mathbb{1}_{\{x\}}(y)\big)$

$$L = \begin{pmatrix} -\lambda & \lambda & & 0 \\ & \ddots & \ddots & \\ & & & \lambda \\ 0 & & & -\lambda \end{pmatrix}$$

representation of rates: $1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 4 \xrightarrow{\lambda} 5 \quad \cdots$

- If $(X_t)_{t \geq 0}$ is a uniform MC with subordinated trans. matrix $\hat{P}$, then $L(x,y) = \lambda\left(\hat{P}_{x,y} - \mathbb{1}_{\{x\}}(y)\right)$ or shortly $L = \lambda(\hat{P} - Id)$

Forward and backward Kolmogorov equation

Let $(P_t)_{t \geq 0}$ the semigroup of a t-MC $(X_t)_{t \geq 0}$.

From the semigroup property, $P_{t+h} = P_t \cdot P_h \quad \forall h,t \geq 0$, $P_0 = Id$, we get

evolve up to t, and then increment of h

- $$\frac{P_{t+h} - P_t}{h} = P_t \cdot \frac{(P_h - Id)}{h} \xrightarrow[h \to 0]{} P_t \cdot L$$

increment of h, and then evolve for time t

- $$\frac{P_{t+h} - P_t}{h} = \frac{(P_h - Id)}{h} \cdot P_t \xrightarrow[h \to 0]{} L P_t$$

$$\Rightarrow \boxed{\frac{d}{dt} P_t = P_t L = L P_t}$$

Kolmogorov forward / backward equations

forward $\qquad$ backward

or equivalently, for all $x, y \in S$: $\frac{d}{dt} P_t(x,y) = P_t L(x,y) = L P_t(x,y)$.

Note: $P_t 1 = 1 \xRightarrow{\text{by backward Kolmogorov's eq.}} L 1 = 0$

conservation of mass $\qquad$ rows of L sum to 0

Theorem [generator]
Let $q(x,y) \geq 0$ $\forall\, x \neq y \in S$ and s.t. the assumptions (A) hold. Then the generator L associated to q is the generator of a unique t-MC $(X_t)_{t \geq 0}$.

<u>Proof</u>: "The idea from $(*)$ is that: $L = \frac{d}{dt} P_t \big|_{t=0} \Longrightarrow P_t = e^{tL}$."

<u>1° STEP</u>: Construct a t-MC with generator L.

Define: $\hat{P}_{xy} = \begin{cases} \dfrac{q(x,y)}{\lambda} & \text{if } x \neq y \\ 1 - \sum\limits_{z \neq x} \dfrac{q(x,z)}{\lambda} & \text{if } x = y \end{cases}$, where $\lambda = \sup\limits_{x \in S} \lambda_x$

and notice that $\hat{P}$ is a stochastic matrix $\left(\begin{array}{l} \hat{P}_{x,y} \in [0,1] \\ \sum_y \hat{P}_{x,y} = 1, \ \forall x \in S \end{array} \right)$

Moreover, $L = \lambda (\hat{P} - Id)$.

We consider a MC $(\hat{X}_n)$ with trans. matrix $\hat{P}$, an independent $PP(\lambda)$, $(N_t)_{t \geq 0}$, and set $X_t := \hat{X}_{N_t}$, $\forall t \geq 0$.

Then $(X_t)_{t \geq 0}$ is a uniform MC (and hence a t-MC) s.t.

$$P_t = e^{\lambda t (\hat{P} - Id)} = e^{tL} \Longrightarrow \frac{d}{dt} P_t \Big|_{t=0} = L. \quad \#$$

2° STEP: Uniqueness

Assume that there exists a t-MC $(\tilde{X}_t)_{t \geq 0}$ with semigroup $(\tilde{P}_t)_{t \geq 0}$ and same generator L $\left(\lim\limits_{s \downarrow 0} \frac{1}{s} (\tilde{P}_s - Id) = L \right)$.

From the forward Kolmogorov's equation, we can write:

$$\underbrace{P_t(x,y) - \tilde{P}_t(x,y)}_{=:\, r_t(x,y)} = \int_0^t \underbrace{L(P_s - \tilde{P}_s)(x,y)}_{=\, L r_s(x,y)} \, ds$$

$$\Longrightarrow \sup_{s \leq t} \|r_s\|_2 \leq \sup_{s \leq t} \int_0^s \|L \cdot r_u\|_2 \, du \leq \overbrace{\|L\|_2}^{\leq 2\lambda} \cdot \sup_{s \leq t} \int_0^s \|r_u\|_2 \, du$$

$$\leq 2\lambda t \cdot \sup_{s\leq t} \|r_s\|_2 < \sup_{s\leq t} \|r_s\|_2 \quad \#$$

taking $t < \frac{1}{2\lambda}$ and if $\sup_{s\leq t} \|r_s\|_2 > 0$

$\Longrightarrow r_s = 0 \quad \forall s \leq t < \frac{1}{2\lambda}$

$\Longrightarrow P_s = \tilde{P}_s \quad \forall s \leq t < \frac{1}{2\lambda}$

Iterating over t $(2t, 3t, \dots)$, we get $P_t = \tilde{P}_t \quad \forall t \geq 0$. $\#$

Consequences: (under assumptions A)

1. A t-MC is uniquely determined by its generator L, i.e.

$$P_t = e^{tL}$$

2. Following the proof, a t-MC can be constructed as a uniform MC built over
   - a subordinated MC $(\hat{X}_n)_{n\in\mathbb{N}}$ $\quad \vec{P}_{xy} = \frac{q_{xy}}{\lambda}$, if $x \neq y$
   - a Poisson clock $(N_t)_{t\geq 0}$ $\quad PP(\lambda)$

   independent

There is an equivalent representation of (X_t) which is more convenient in application because it allows to couple $\hat{X}$ and $\hat{N}$ in a unique object and holds even when $\lambda = +\infty$.

Transition times and the embedded MC

For a t-MC $(X_t)_{t\geq 0}$, consider the transition times $(\tau_n)_{n\geq 0}$ s.t

$$\tau_0 = 0 \quad , \quad \tau_n = \inf\{t > \tau_{n-1} : X_t \neq X_{\tau_{n-1}}\} \quad \forall n \geq 1$$

so that $\quad X_t = X_{\tau_{n-1}} \quad \forall t \in [\tau_{n-1}, \tau_n)$

Remark: from assumptions A (non-explosion), $\tau_n \xrightarrow[n\to\infty]{} +\infty$, i.e there is not a "last jump" in a finite time.

Given a t-MC $(X_t)_{t\geq 0}$, let $\tilde{X}_n = X_{\tau_n}$, $\forall n \geq 0$.

Theorem [embedded MC]:

In the above setting, it holds:

a. $(\tilde{X}_n)_{n\geq 0}$ is a MC on S, called embedded MC, with trans. prob: $\tilde{P}_{x,y} = \dfrac{q(x,y)}{\lambda_x}$, $\forall y \neq x$ (and 0 if $y=x$)

b. Given $(\tilde{X}_n)_{n\geq 0}$, the transition time increments $(\tau_{n+1}-\tau_n)_{n\geq 0}$ are all independent and s.t.

$$\mathcal{L}aw\left(\tau_{n+1}-\tau_n \mid \tilde{X}_n = x\right) = Exp(\lambda_x), \quad \forall x \in S$$

Proof (ideas):

a. $(\tilde{X}_n)$ can be constructed from the subordinated MC $(\hat{X}_n)$, by neglecting the self-jump to a state (so that $\tilde{P}_{x,x} = 0$ $\forall x \in S$). Hence it inherits the Markov property from $(\hat{X}_n)$ (verify!), and for $x \neq y$ we get

$$\tilde{P}_{x,y} = \mathbb{P}\left(\tilde{X}_n = y \mid \tilde{X}_{n-1} = x\right) = \sum_{m=0}^{\infty} (\hat{P}_{x,x})^m \cdot \hat{P}_{x,y} = \frac{\hat{P}_{x,y}}{1-\hat{P}_{x,x}}$$

$$= \frac{q(x,y)}{\lambda_x} \quad \leftarrow \left(\hat{P}_{x,y} = \frac{q(x,y)}{\lambda}, \; \hat{P}_{x,x} = 1 - \frac{\lambda_x}{\lambda} \right)$$

#

b. We first consider τ_1 and assume that $X_0 = \tilde{X}_0 = x$. Then, $\forall t>0$

$$\mathbb{P}_x(\tau_1 > t) = \mathbb{P}_x\Big(\overset{\text{disjoint}}{\bigcup_{k=0}^{\infty}} \{ N_t = k, \; \hat{X}_j = x \;\; \forall j=1,\dots,k \} \Big)$$

$$\mathbb{P}_x(\tau_1 > t) = \mathbb{P}_x\Big(\bigcup_{k=0}^{\infty} \{ N_t = k,\ \hat{X}_j = x\ \forall j = 1,\dots,k \} \Big)$$

(N_t) is $PP(\lambda)$, $(\hat{X}_m)$ subordinated MC

i indep.

$$= \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \left(1 - \frac{\lambda_x}{\lambda}\right)^k = e^{-\lambda t} e^{(\lambda - \lambda_x)t} = e^{-\lambda_x t}$$

$$\Rightarrow \quad \mathrm{Law}(\tau_1 \mid \tilde{X}_0 = x) = \mathrm{Exp}(\lambda_x)$$

Using the strong Markov property of $(\tilde{X}_m)$ and independence of the inter-arrivals of (N_t), one can prove similarly that

$$\mathrm{Law}(\tau_{m+1} - \tau_m \mid \tilde{X}_m = x) = \mathrm{Exp}(\lambda_x)$$

and the independence of $(\tau_{m+1} - \tau_m)_{m \geq 0}$, given $(\tilde{X}_m)$. #

Graphical construction of t-MC π

The previous theorem suggests the following graphical construction, which is the most common in application:

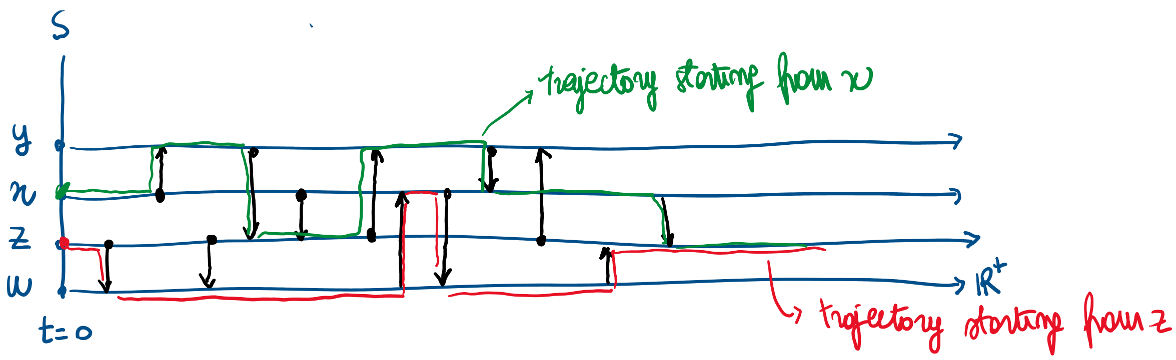
1. $\forall x \in S$, let us consider a $PP(\lambda_x)$, denoted $N^x = (N^x_t)_{t \geq 0}$, so that N^x, $x \in S$ are all independent.

2. Each arrival of N^x corresponds to a jump in a site $y \neq x \in S$ (mark of the arrival) with probability $\tilde{P}_{x,y} = \frac{q(x,y)}{\lambda_x}$ $\Rightarrow$ the process N^x can be seen as a marked PP, and decomposed as $N^x_t = \sum_{y \neq x} N^{x,y}_t$,

   where $N^{x,y}_t$ = # of transition $x \rightsquigarrow y$ in $[0,t]$ is $PP(q(x,y))$.

Graphically, for each state $x \in S$, we sample N^x, and draw an arrow among x and y at each arrival of $N^{x,y}$ $(\forall y \in S)$.

Then we get a trajectory starting from a state (say x) at $t=0$, Then we get a trajectory starting from a state (say x) at $t=0$, and following the time line and the arrows as in the picture:



Remark: this construction holds also thanks to the absence of memory of the exponential law, so that we can initiate all the PP's $(N_x)_{x\in S}$ at $t=0$, instead of resetting the time after every transition.

Conclusion: $\begin{cases} q_{x,y} = \text{rate of the Exponential time to jump from } x \text{ to } y \\ \qquad \text{(jump rate)} \\ \lambda_x = \text{rate of the Exponential holding time in } x \\ \qquad \text{(holding rate)} \end{cases}$

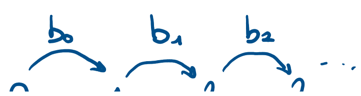
[Indeed, if $T_{x,y} \sim Exp(q_{x,y})$, $\forall y \neq x \in S$, independent, then $T_x = \min_{y\neq x} \{T_{x,y}\} \sim Exp\left(\sum_{y\neq x} q_{x,y}\right) = Exp(\lambda_x)$ is precisely the time spent in x before jumping to a new state.]

Graphical construction of t-MC: Examples

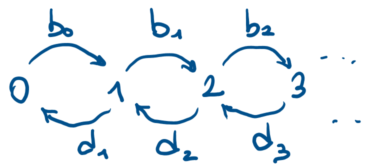
Example 1: Birth and death processes

Consider a t-MC $(X_t)_{t\geq 0}$ on $\mathbb{N}_0$ with jump rates

$$q(x,y) = \begin{cases} b_x & \text{if } y = x+1 \\ d_x & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } \underset{\substack{\uparrow \\ \text{birth rate} \\ \text{at } x}}{b_x}, \underset{\substack{\uparrow \\ \text{death rate} \\ \text{at } x}}{d_x} > 0$$



which can be sketched:

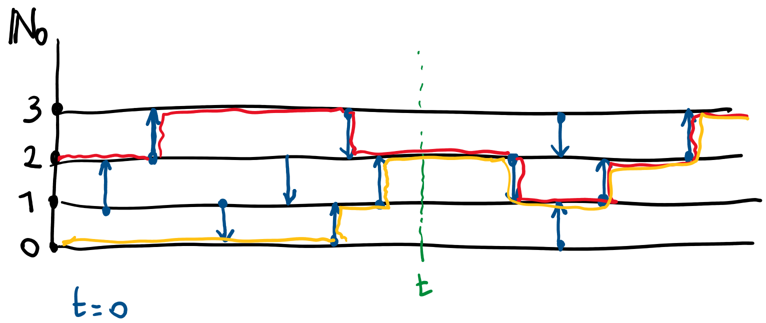


Then:

$\lambda_x = b_x + d_x \quad \forall x \in \mathbb{N}_0 \quad (\text{with } d_0 = 0)$

The embedded MC $(\tilde{X}_n)$ has trans prob $\tilde{P}_{x,y} = \begin{cases} \frac{b_x}{\lambda_x} & \text{if } y = x+1 \\ \frac{d_x}{\lambda_x} & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}$

* Graphical construction



trajectories starting from 0, and 2

Interpretation: They model the evolution of the size of a population (hence the name) but they are also used to describe queuing system, where "birth" = "arrival of a client", "death" = "end service of a client".

Special cases:

* if $b_x = b$ and $d_x = 0$, $\forall x \in \mathbb{N}_0 \Rightarrow PP(b)$
* if $d_x = 0 \ \forall x \in \mathbb{N}_0 \Rightarrow$ pure birth process = counting process with $Exp(b_x)$ arrival times
* if $b_x = b$, and $d_x = d$, $\forall x \in \mathbb{N}_0 \Rightarrow$ queue system M/M/1
  (1 server; Markovian arrivals at rate b; Markovian service at rate d)

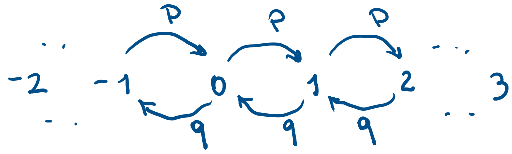
Example 2: Continuous time RW on $\mathbb{Z}$ → or on a graph $G = (V, E)$

Consider a t-MC $(X_t)_{t \geq 0}$ on $\mathbb{Z}$ with jump rates

$$q(x,y) = \begin{cases} p & \text{if } y = x+1 \\ q & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } p, q \geq 0$$

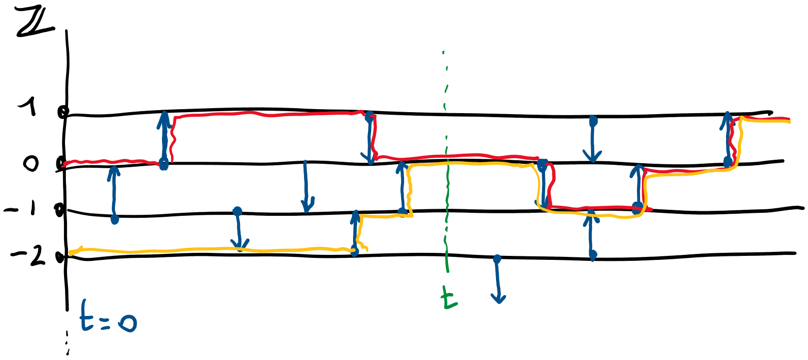
$\begin{cases} q & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}$

which can be sketched:



Then:

- $\lambda = \lambda_x = p + q \quad \forall x \in \mathbb{Z}$
- The embedded MC $(\tilde{X}_n)$ has trans prob $\tilde{P}_{xy} = \begin{cases} \frac{p}{p+q} & \text{if } y = x+1 \\ \frac{q}{p+q} & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}$
- Graphical construction



At each $x \in \mathbb{Z}$, there are indep. PP(p) and PP(q): at each arrival we insert a directed arrow