ADVANCED STOCHASTIC PROCESSES - 15° LECTURE

- Random measure m is characterized by its Laplace functional
$$\mathcal{L}_m(f) = \mathbb{E}\left(e^{-m(f)}\right), \text{ for } f: E \to \mathbb{R}^+, f \in \mathcal{E}$$

Theorem [Characterization of PPP(Λ)]

Let N be a random measure on $(E, \mathcal{E})$. Then N is a PPP(Λ)

$$\iff \quad \mathcal{L}_N(f) = e^{-\Lambda(1-e^{-f})}, \quad \forall f \in \mathcal{E}, f \geq 0 \quad (*)$$

where $\Lambda(1-e^{-f}) = \int_E \left(1-e^{-f(x)}\right) \Lambda(dx)$

Proof: $\Longrightarrow$:

1° First we verify $(*)$ for function $f = a \cdot \mathbb{1}_A$, with $A \in \mathcal{E}$, $a \geq 0$:

$N(a\mathbb{1}_A) = \int_E a \mathbb{1}_A(x)\, N(dx) = a\, N(A)$, hence

$$\mathbb{E}\left(e^{-aN(A)}\right) = \sum_{j=0}^{\infty} e^{-a \cdot j} \cdot \frac{(\Lambda(A))^j}{j!} e^{-\Lambda(A)} = e^{-\Lambda(A)\left(1-e^{-a}\right)}$$

check!
$$= e^{-\Lambda\left(1-e^{-a\mathbb{1}_A}\right)} \quad \checkmark$$

2° Then we verify $(*)$ for simple functions $f = \sum_{i=1}^{m} a_i \cdot \mathbb{1}_{A_i}$,
where $A_1, \dots, A_m \in \mathcal{E}$ are disjoint, $a_1, \dots, a_m \geq 0$:

$N(f) = \sum_{i=1}^{m} a_i N(A_i)$, where $N(A_i)$ are indep. by def. of PPP(Λ).

Hence: $$\mathbb{E}\left(e^{-N(f)}\right) = \prod_{i=1}^{m} \mathbb{E}\left(e^{-a_i N(A_i)}\right) = e^{\left(-\sum_{i=1}^{m} \Lambda(A_i)(1-e^{-a_i})\right)}$$
$$= e^{-\Lambda(1-e^{-f})} \quad \checkmark$$

3° We conclude the proof for general functions $f \in \mathcal{E}$, $f \geq 0$.

Consider $(f_n)_{n \in \mathbb{N}}$ simple functions s.t. $f_n \nearrow f$.

...tone convergence

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By the continuity of the Laplace functionals:

$$\mathbb{E}\left(e^{-N(f)}\right) = \lim_{n\to\infty} \mathbb{E}\left(e^{-N(f_n)}\right) = \lim_{n\to\infty} e^{-\Lambda\underbrace{(1-e^{-f_n})}_{\text{increasing to } 1-e^{-f}}} \overset{\text{monotone convergence theorem}}{=} e^{-\Lambda(1-e^{-f})} \quad \checkmark$$

$\Leftarrow)$ follows from Proposition on the Laplace functional, together with the preceding computation. #

Remark: Consider Example 1 with

$$m(A) = \sum_{k=1}^{Y} \mathbb{1}_A(X_k) \qquad \text{for } \left.\begin{array}{l} Y \sim \mathrm{Poi}(\lambda) \\ (X_k)_{k\in\mathbb{N}} \text{ iid} \sim \nu \end{array}\right\} \text{ indep.}$$

We have shown that

$$\mathcal{L}_m(f) = e^{-\lambda\nu(1-e^{-f})} \qquad (\lambda\in\mathbb{R}^+,\ \nu\in\mathcal{P}(E,\mathcal{E}))$$

By the Theorem above, $m \sim \mathrm{PPP}(\Lambda)$, with $\Lambda = \lambda\nu$.

This provides a general idea to construct $\mathrm{PPP}(\Lambda)$ as shown by the next result.

Theorem: For any $\Lambda \in \mathcal{M}(E,\mathcal{E})$, there exists a $\mathrm{PPP}(\Lambda)$.

Proof

1. Assume that $\Lambda(E) < \infty$ and define $\nu \in \mathcal{P}(E,\mathcal{E})$ s.t.

$$\nu(A) := \frac{\Lambda(A)}{\Lambda(E)}, \quad \forall A \in \mathcal{E}$$

Then consider $(X_k)_{k\in\mathbb{N}}$ i.i.d $\sim \nu$ and indep. of $Y \sim \mathrm{Poi}(\Lambda(E))$ and define

$$m(A) := \sum_{k=1}^{Y} \mathbb{1}_A(X_k), \quad \forall A \in \mathcal{E}$$

We have already shown that m is a random measure with Laplace functional

$$\mathcal{L}_m(f) = e^{-\Lambda(E)\cdot\nu(1-e^{-f})} \overset{\nu(\cdot)=\frac{\Lambda(\cdot)}{\Lambda(E)}}{=} e^{-\Lambda(1-e^{-f})}$$

By the previous Theorem [Characterization of PPP(Λ)], it turns out that $m \sim PPP(\Lambda)$ as wanted.

2. If $\Lambda(E) = \infty$, we can use the fact that Λ is σ-finite, and hence that exist $(E_n)_{n\geq 0}$ s.t. $E_n \uparrow E$ with $\Lambda(E_n) < \infty$.

Define: $\begin{cases} \Lambda^1(A) := \Lambda(E_1 \cap A) \\ \Lambda^n(A) := \Lambda((E_n \setminus E_{n-1}) \cap A) \quad \forall\, n \geq 2 \end{cases}, \quad \forall A \in \mathcal{E}$

Then Λ^n is finite, and by step 1. we can construct $m^n \sim PPP(\Lambda_n)$. independently, and set $m := \sum_{n=1}^{\infty} m^n \implies m \sim PPP(\Lambda)$

#

<u>Corollary</u> [Properties of PPP(Λ)]

1. If $f \in L^1(\Lambda) \implies \mathbb{E}(N(f)) = \Lambda(f)$

$\longrightarrow$ for $f = \mathbb{1}_A$, we get $\mu(A) = \mathbb{E}(N(A)) = \Lambda(A)$ $\left(\text{mean} = \text{intensity}\right)$

2. If $f \in L^2(\Lambda) \cap \underline{L^1(\Lambda)}_{*} \implies Var(N(f)) = \Lambda(f^2)$

$\longrightarrow$ for $f = \mathbb{1}_A$, we get $Var(N(A)) = \Lambda(A)$

* not obvious if Λ is not a finite measure

Verify! The proof is an application of the previous Theorem.

Example

1. <u>Particles in boxes:</u>

Let E countable, $\mathcal{E} = \mathcal{P}(E)$ and $\Lambda \in M_+(E)$

1. Particles in boxes:

- Let E countable, $\mathcal{E} = \mathcal{P}(E)$, and $\Lambda \in \mathcal{M}(E)$

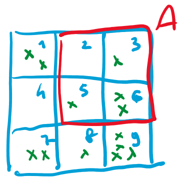
  ↳ collection of boxes of $\mathbb{R}^d$

- Consider independent r.v. $(W_x)_{x \in E}$ s.t. $W_x \sim \text{Poi}(\Lambda(\{x\}))$

  ↳ number of particles in the box x

Then $N(A) := \sum_{x \in E} W_x \underbrace{\mathbb{1}_A(x)}_{= \delta_x(A)}$, $\forall A \in \mathcal{E}$

is a $PPP(\Lambda)$ describing the particles in boxes



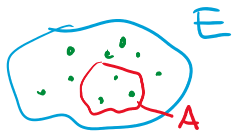
2. Stones in a field:

- Let $Y \sim \text{Poi}(\lambda)$ → number of stones

- Let $(X_n)_{n \in \mathbb{N}}$ iid on $(E, \mathcal{E})$, with law ν, independent of Y

  ↳ each stone is thrown, and takes a position in E, independently of the other, described by X_n (for stone n)

Then $N(A) = \sum_{n=1}^{Y} \mathbb{1}_A(X_n)$, $\forall A \in \mathcal{E}$



is a $PPP(\lambda \cdot \nu)$ describing the configuration of the thrown stones in E. Indeed, this corresponds to the example 1 above where we obtained (with updated notation)

$$\mathcal{L}_N(f) = e^{-\lambda(1 - \nu(e^{-f}))} = e^{-\lambda\nu(1-e^{-f})}$$

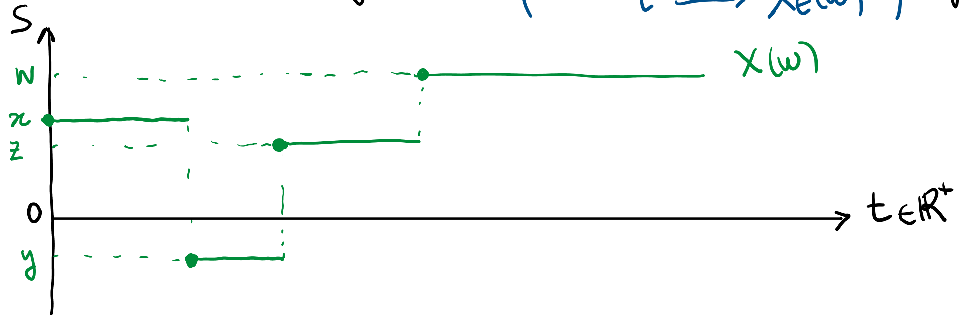
[verify! → $\lambda \in \mathbb{R}^+$, $\nu \in \mathcal{M}(E)$]

Markov chains in continuous time (t-MC)

Intro: Let S be a countable space. We want to define on S an analogous of MC but in continuous time → $(X_t)_{t \geq 0}$.

Since the space is discrete, the process moves by jumping among the states

Since the space is discrete, the process moves by jumping among the states of S after having spent there a "random holding time", hence realizing piecewise constant trajectories $\left(\begin{array}{l} X(\omega): \mathbb{R}^+ \longrightarrow S \\ \quad\quad\;\; t \longrightarrow X_t(\omega) \end{array} \right)$ of this type:



Def: $(X_t)_{t\geq 0}$ is a Markov Process on S or <u>continuous time MC</u>, if, $\forall y \in S$ and $\forall s,t \geq 0$, it holds

$$\mathbb{P}(X_{s+t} = y \mid \mathcal{F}_s) = \mathbb{P}(X_{s+t} = y \mid X_s) \quad \text{(Markov property)}$$

In particular, <u>X is homogeneous</u> if

$$\mathbb{P}(X_{s+t} = y \mid X_s = x) = \mathbb{P}(X_t = y \mid X_0 = x) =: \underline{P_t(x,y)} \quad \text{(transition probability)}$$

Remark: By definition, $\forall x, y \in S$ and $t \geq 0$, it holds

- $P_t(x,y) \geq 0$
- $\sum_{y \in S} P_t(x,y) = 1$

Clearly, $(P_t(x,y))$ generalizes (to cont. times) the n-step trans. prob. $(P^n_{x,y})$ of MC. But while these last ones are identified as powers of P, in continuous time we will rather use the idea of "transition rates".

Notation:

1. <u>The law of $(X_t)_{t\geq 0}$</u> s.t. $X_0 \sim \nu$, with $\nu \in \mathcal{P}(S)$ is denoted $\mathbb{P}_\nu$ and characterized over cylinder sets so that for $m \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_m$, $\forall x_1, \dots, x_m \in S$:

for $m \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_m$, $\forall\, x_1, \dots, x_m \in S$:

$$\mathbb{P}_\nu \left(X_{t_0} = x_0, X_{t_1} = x_1, \dots, X_{t_m} = x_m \right) = \nu(x_0) \prod_{j=1}^{m} P_{t_j - t_{j-1}}(x_{j-1}, x_j)$$

In particular $\mathbb{P}_\nu \left(X_t = y \right) = \sum_{x \in S} \nu(x) P_t(x,y)$, and if $\nu = \delta_x$ then $\mathbb{P}_x \left(X_t = y \right) = P_t(x,y)$.

2. For $t \geq 0$, we let P_t be the operator $P_t : \mathbb{R}^S \to \mathbb{R}^S$ s.t. $f \to P_t f$

$$\underline{P_t f(x)} = \sum_{y \in S} P_t(x,y) f(y) \equiv \mathbb{E}_x \left(f(X_t) \right) \quad \text{(Average value of } f(X_t))$$

and also $P_t : \mathcal{P}(S) \to \mathcal{P}(S)$ s.t. $\mu \longrightarrow \mu P^t$

$$\underline{\mu P_t(y)} = \sum_{x \in S} \mu(x) P_t(x,y) \equiv \mathbb{P}_\mu \left(X_t = y \right) \quad \text{(law of } X_t \mid X_0 = \mu)$$

Then $(P_t)_{t \geq 0}$ is the semigroup associated to $(X_t)_{t \geq 0}$, and by its definition (and the Markov property):

$$\begin{cases} * \ P_{t+s} = P_t \cdot P_s, & \forall\, s,t \geq 0 \\ * \ P_0 = Id & \end{cases}$$

Example 1: A (homogeneous) $PP(\lambda)$ $(N_t)_{t \geq 0}$ is a t-MC.

Indeed, for $s_1 < s_2 < \dots < s_k$ and $i_1 \leq i_2 \leq \dots \leq i_k$, let $H = \{ N_{s_1} = i_1, \dots, N_{s_k} = i_k \}$ and for $s > s_k$ and $j \geq i$:

$$\mathbb{P}\left(N_{t+s} = j \mid N_s = i, H \right) = \mathbb{P}\left(N_{t+s} - N_s = j - i \mid N_s = i, H \right)$$

indep. and stat. of increments

$$\underset{\text{Markov property}}{=} \mathbb{P}\left(N_t = j - i \right) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t} \underset{\text{trans. probabilities}}{=} P_t(i,j)$$

Example 2: Let $(\hat{X}_n)_{n \geq 0}$ be a MC on S with trans. matrix $\hat{P}$, and let $(N_t)_{t \geq 0}$ be a $PP(\lambda)$ indep. of $\hat{X}$, with arrival times $(T_k)_{k \geq 1}$.

Let $(N_t)_{t\geq 0}$ be a $PP(\lambda)$ indep. of $\hat{X}$, with arrival times $(T_k)_{k\geq 1}$.

Define $\underline{X_t := \hat{X}_{N_t}}$ $\forall\, t\geq 0$.

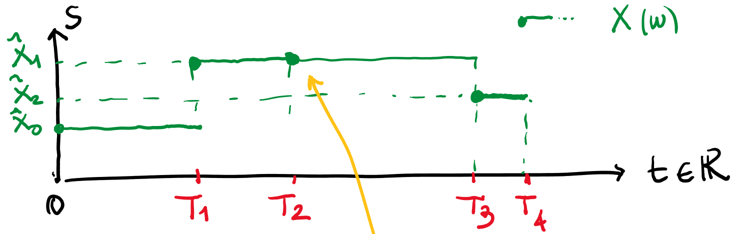
Then $(X_t)_{t\geq 0}$ is a t-MC called uniform MC s.t

$$P_t(x,y) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \hat{P}^{(n)}_{x,y} \quad \text{or} \quad \boxed{P_t = e^{-\lambda t (I_d - \hat{P})}}$$

$\hat{X}_n$ is called subordinated chain

(N_t) is called Poissonian clock.

Graphically:



Note: • $X_{T_n} = \hat{X}_n$

• jumps occur only at the arrival times T_k's, but there may be arrival times where the t-MC does not jump, as the MC $\hat{X}$ may jump on the same state.