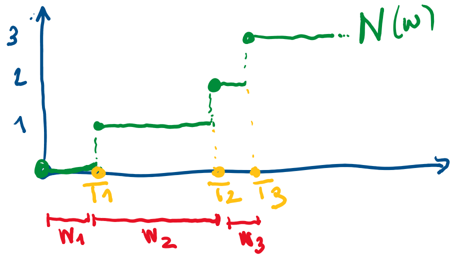
ADVANCED STOCHASTIC PROCESSES - 14° LECTURE

SUMMARY: "Equivalent representations of PP on $\mathbb{R}^+$"

We have seen that a $PP(\lambda)$, $(N_t)_{t\geq 0}$, can be thought in different ways:

1. As a trajectory $N=(N_t)_{t\geq 0}$ represented ($\forall\, \omega\in\Omega$) as an integer-valued step function;



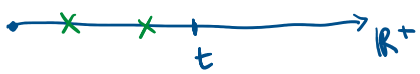
2. A sequence of points $(T_n)_{n\geq 0}$ in $\mathbb{R}^+$ with $T_n := \inf\{t\geq 0 : N_t = n\}$
3. A sequence of time-intervals $(W_n)_{n\in\mathbb{N}}$ s.t. $W_n = T_n - T_{n-1}$

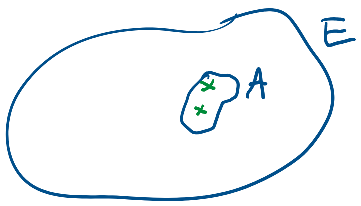
While repr. 1. and 3. are specific to time-evolution in $\mathbb{R}^+$, the the repr. 3. admits generalizations $\longrightarrow$ main idea is to construct random set of points on more general spaces

($\mathbb{R}^+ \longrightarrow$ e.g. $\mathbb{R}$, $\mathbb{R}^d$ or general metric space $(E,\mathcal{E})$ which is complete, separable, locally compact)
$\hookrightarrow$ Polish space

MAIN IDEA:

$$N_t = \#\text{ arrivals in } [0,t] = \#\text{ points in } [0,t] \xrightarrow{\text{generalized to}} N(A) = \#\text{ points in } A \quad \text{for } A\in\mathcal{E}$$





Def [Poisson Point Process]

Let Λ be a σ-finite measure on $(E,\mathcal{E})$ (shortly, $\Lambda\in\mathcal{M}(E,\mathcal{E})$)

A Poisson Point Process with intensity Λ ($PPP(\Lambda)$) is a function

$$N : \Omega\times\mathcal{E} \longrightarrow \overline{\mathbb{N}}_0 \qquad \text{such that } \forall\, k\in\mathbb{N},\ \forall\, m_1,\dots,m_k\in\mathbb{N}_0,$$
$$(\omega, A) \longrightarrow N_\omega(A) \qquad \forall \text{ disjoints } A_1,\dots,A_k\in\mathcal{E}:$$

$(\omega, A) \longrightarrow N_\omega(A)$ $\forall$ disjoints $A_1, \dots, A_K \in \mathcal{E}$:

$$\mathbb{P}\left(N(A_i) = n_i, \ \forall i = 1, \dots K\right) = \prod_{i=1}^{K} \frac{\Lambda(A_i)^{n_i}}{n_i!} e^{-\Lambda(A_i)}$$

Interpretation: $N_\omega(A) = \#$ points in A (for the realization ω)

Remarks:

- If $E = \mathbb{R}^d$ (or subsets of it) and $\Lambda = \lambda \cdot \mathrm{Leb}_{\mathbb{R}^d}$, with $\lambda > 0$, we obtain the precise generalization of homogeneous $PP(\lambda)$ on $\mathbb{R}^+$.
  In this case: $\Lambda(A) = \lambda \,\mathrm{Vol}(A)$
- The definition implicitly uses that $\forall A \in \mathcal{E}$
  $N(A): \Omega \to \overline{\mathbb{N}}$ is a r.v with law $\mathrm{Poi}(\Lambda(A))$

POISSON POINT PROCESS versus POISSON RANDOM MEASURE

A countable set of points $(x_i)_{i \in I} \subseteq E$ can be put in correspondence with a counting measure μ on $(E, \mathcal{E})$, by setting

$$\mu(A) = \sum_{i \in \mathbb{N}} \delta_{x_i}(A) = \# \text{ of points in } A$$

In this sense, the $PPP(\Lambda)$ can be considered as a random counting measure on $(E, \mathcal{E})$. Let us formalize this idea.

Def [random measure]: Consider $(\Omega, \mathcal{F}, \mathbb{P})$ prob. space and $(E, \mathcal{E})$.
A random measure m on $(E, \mathcal{E})$ is a function $m: \Omega \times \mathcal{E} \to \overline{\mathbb{R}}_+$
$(\omega, A) \to m_\omega(A)$
such that:

* $m(A): \Omega \to \overline{\mathbb{R}}_+$ is a r.v. $\forall A \in \mathcal{E}$
* $m_\omega: \mathcal{E} \longrightarrow \overline{\mathbb{R}}_+$ is a measure $\forall \omega \in \Omega$

Moreover, it is a random counting measure if

Moreover, it is a random counting measure if

$$m_\omega := \sum_{i \in \mathbb{N}} \delta_{X_i(\omega)} \quad , \text{ for } (X_i)_{i\in\mathbb{N}} \text{ set of random point}$$

__Notation:__

If $f : E \to \mathbb{R}_+$ and m is a random measure on $(E, \mathcal{E})$, then:

$$m_\omega(f) := \int_E m_\omega(dx)\, f(x) \overset{\text{if counting measure}}{=} \sum_{i\in\mathbb{N}} f(X_i(\omega)) \qquad \left(\begin{array}{c}\text{average of } f \\ \text{w.r.t. } m_\omega\end{array}\right)$$

$\longrightarrow$ $m(f) : \Omega \to \bar{\mathbb{R}}_+$ is a r.v. with mean $\int_\Omega \mathbb{P}(d\omega)\, m_\omega(f) \overset{\text{if counting meas.}}{=} \sum_{i\in\mathbb{N}} \mathbb{E}(f(X_i))$

In particular, taking $f = \mathbb{1}_A$ for $A \in \mathcal{E}$, $m_\omega(\mathbb{1}_A) = m_\omega(A)$,

we define $\boxed{\mu(A) := \mathbb{E}(m(A)) = \int_\Omega \mathbb{P}(d\omega)\, m_\omega(A)}$, $\forall A \in \mathcal{E}$

and notice that $\mu : \mathcal{E} \to \bar{\mathbb{R}}^+$ is a measure on $(E, \mathcal{E})$

called mean of the random measure m (called mean or intensity measure)

__Example 1:__ Let $(X_k)_{k\in\mathbb{N}}$ i.i.d r.v. with common law ν on $(E, \mathcal{E})$, and let $Y \sim \text{Poi}(\lambda)$, independent of the X_k s.

* Define the random measure $m(A) = \sum_{k=1}^{Y} \delta_{X_k}(A)$, $\forall A \in \mathcal{E}$

$$\left[\begin{array}{ll} \bullet\ m(A) : \Omega \to \mathbb{R}_+ \quad \omega \to m_\omega(A) & \text{is a r.v. } \forall A \in \mathcal{E} \\ \bullet\ m_\omega : \mathcal{E} \to \mathbb{R}^+ \quad A \to m_\omega(A) & \text{is a measure, } \forall \omega \in \Omega \end{array}\right]$$

We compute its intensity measure μ:

$$\mu(A) = \mathbb{E}(m(A)) = \mathbb{E}(\mathbb{E}(m(A)|Y)) = \sum_{j=1}^{\infty} \mathbb{P}(Y=j) \cdot j \cdot \underbrace{\mathbb{P}(X_k \in A)}_{= \nu(A)}$$

$$= \nu(A) \cdot \mathbb{E}(Y) = \lambda \cdot \nu(A)$$

$\longrightarrow$ hence $\mu : \mathcal{E} \to \mathbb{R}_+$
$A \to \lambda \nu(A)$

$\rightarrow$ hence $\mu : \mathcal{E} \longrightarrow \mathbb{R}_+$
$A \rightarrow \lambda \cdot \nu(A)$

The main tool to characterize the law of $m_{(\cdot)}$ is the Laplace functional, as provided by the next Proposition (without proof!)

Proposition: The law of $m_{(\cdot)}$ is uniquely determined by the

$$\mathcal{L}_m(f) := \mathbb{E}\left(e^{-m(f)}\right), \quad \forall\, f : E \to \mathbb{R}^+,\ f \in \mathcal{E}$$

where $f \longrightarrow \mathcal{L}_m(f)$ is called Laplace functional.

Continuity property: if $(f_n) \uparrow f \Rightarrow \mathcal{L}_m(f_n) \xrightarrow[n\to\infty]{} \mathcal{L}_m(f)$

Note: 1. the Laplace functional of a random measure m, plays the role of a generating function for r.v.

2. the same characterization holds true for the characteristic functional $\varphi_m(f) = \mathbb{E}\left(e^{im(f)}\right)$, for $f : E \to \mathbb{R}$, $f \in \mathcal{E}$ and bounded.

Example 1 (continuation): let $m(A) = \sum_{k=1}^{Y} \mathbb{1}_A(X_k)$,

for $(X_k)_{k\in\mathbb{N}}$ i.i.d $\sim \nu$ and indep. of $Y \sim \mathrm{Poi}(\lambda)$, and $f \geq 0$, $\mathcal{E}$-meas.

$$\mathcal{L}_m(f) = \mathbb{E}\left(e^{-m(f)}\right) = \mathbb{E}\left(\prod_{k=1}^{Y} e^{-f(X_k)}\right) = \mathbb{E}\left(\mathbb{E}\left(\text{“ ”} \mid Y\right)\right)$$

$$= \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} \underbrace{\mathbb{E}\left(e^{-f(X_k)}\right)}_{\int_E e^{-f(x)}\nu(dx) = \nu(e^{-f})}{}^{j} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \nu(e^{-f})^j$$

$$= e^{-\lambda(1-\nu(e^{-f}))} = e^{-\lambda\nu(1-e^{-f})}$$

- Random measure m is characterized by its Laplace functional

$$\mathcal{L}_m(f) = \mathbb{E}\left(e^{-m(f)}\right), \quad \text{for } f : E \to \mathbb{R}^+,\ f \in \mathcal{E}$$

Theorem [Characterization of PPP(Λ)]

Let N be a random measure on $(E, \mathcal{E})$. Then N is a PPP(Λ)

$$\iff \quad \mathcal{L}_N(f) = e^{-\Lambda(1-e^{-f})}, \quad \forall f \in \mathcal{E}, f \geq 0 \quad (*)$$

where $\Lambda(1-e^{-f}) = \int_E (1-e^{-f(x)})\,\Lambda(dx)$

Proof: $\Rightarrow$:

1. First we verify $(*)$ for function $f = a \cdot \mathbb{1}_A$, with $A \in \mathcal{E}$, $a \geq 0$:

$N(a\mathbb{1}_A) = \int_E a\mathbb{1}_A(x)\,N(dx) = a\,N(A)$, hence

$$\mathbb{E}\left(e^{-aN(A)}\right) = \sum_{j=0}^{\infty} e^{-a \cdot j} \cdot \frac{(\Lambda(A))^j}{j!} e^{-\Lambda(A)} = e^{-\Lambda(A)\left(1-e^{-a}\right)}$$

$$\overset{\text{check!}}{=} e^{-\Lambda\left(1-e^{-a\mathbb{1}_A}\right)} \quad \checkmark$$

2. Then we verify $(*)$ for simple functions $f = \sum_{i=1}^{n} a_i \cdot \mathbb{1}_{A_i}$,

where $A_1, \dots, A_n \in \mathcal{E}$ are disjoint, $a_1, \dots, a_n \geq 0$:

$N(f) = \sum_{i=1}^{n} a_i N(A_i)$, where $N(A_i)$ are indep. by def. of PPP(Λ).

Hence: $$\mathbb{E}\left(e^{-N(f)}\right) = \prod_{i=1}^{n} \mathbb{E}\left(e^{-a_i N(A_i)}\right) = e^{\left(-\sum_{i=1}^{n} \Lambda(A_i)(1-e^{-a_i})\right)}$$

$$= e^{-\Lambda(1-e^{-f})} \quad \checkmark$$

3. We conclude the proof for general functions $f \in \mathcal{E}$, $f \geq 0$.

Consider $(f_n)_{n \in \mathbb{N}}$ simple functions s.t. $f_n \nearrow f$.

By the continuity of the Laplace functionals:

$$\mathbb{E}\left(e^{-N(f)}\right) = \lim_{n\to\infty} \mathbb{E}\left(e^{-N(f_n)}\right) = \lim_{n\to\infty} e^{-\Lambda(\underbrace{1-e^{-f_n}}_{\text{increasing to } 1-e^{-f}})} \overset{\text{monotone convergence theorem}}{=} e^{-\Lambda(1-e^{-f})} \quad \checkmark$$

$\Leftarrow$) follows from Proposition on the Laplace functional, together with the preceding computation. #

with the preceding computation. #

Remark: Consider Example 1 with

$$m(A) = \sum_{k=1}^{Y} \mathbb{1}_A(X_k) \qquad \text{for } \left.\begin{array}{l} Y \sim Poi(\lambda) \\ (X_k)_{k\in\mathbb{N}} \text{ iid} \sim \nu \end{array}\right\} \text{indep.}$$

We have shown that

$$\mathcal{L}_m(f) = e^{-\lambda\nu(1-e^{-f})} \qquad (\lambda \in \mathbb{R}^+, \nu \in \mathcal{P}(E,\mathcal{E}))$$

By the Theorem above, $m \sim PPP(\Lambda)$, with $\Lambda = \lambda\nu$.

This provides a general idea to construct $PPP(\Lambda)$ as shown by the next result.

Theorem: For any $\Lambda \in \mathcal{M}(E,\mathcal{E})$, there exists a $PPP(\Lambda)$.

Proof

1. Assume that $\Lambda(E) < \infty$ and define $\nu \in \mathcal{P}(E,\mathcal{E})$ s.t.

$$\nu(A) := \frac{\Lambda(A)}{\Lambda(E)}, \quad \forall A \in \mathcal{E}$$

Then consider $(X_k)_{k\in\mathbb{N}}$ i.i.d $\sim \nu$ and indep. of $Y \sim Poi(\Lambda(E))$ and define

$$m(A) := \sum_{k=1}^{Y} \mathbb{1}_A(X_k), \quad \forall A \in \mathcal{E}$$

We have already shown that m is a random measure with Laplace functional

$$\mathcal{L}_m(f) = e^{-\Lambda(E)\cdot\nu(1-e^{-f})} \overset{\nu(\cdot) = \frac{\Lambda(\cdot)}{\Lambda(E)}}{=} e^{-\Lambda(1-e^{-f})}$$

By the previous Theorem [Characterization of $PPP(\Lambda)$], it turns out that $m \sim PPP(\Lambda)$ as wanted.

2. If $\Lambda(E) = \infty$ we can use the fact that Λ is σ-finite

Then $m_1 \sim PPP(\Lambda)$ as wanted.

2. If $\Lambda(E) = \infty$, we can use the fact that Λ is σ-finite, and hence that exist $(E_n)_{n \geq 0}$ s.t. $E_n \uparrow E$ with $\Lambda(E_n) < \infty$.

Define: $\begin{cases} \Lambda^1(A) := \Lambda(E_1 \cap A) \\ \Lambda^n(A) := \Lambda((E_n \setminus E_{n-1}) \cap A) \quad \forall n \geq 2 \end{cases}, \forall A \in \mathcal{E}$

Then Λ^n is finite, and by step 1. we can construct $m^n \sim PPP(\Lambda_n)$. independently, and set $m := \sum_{n=1}^{\infty} m^n \implies m \sim PPP(\Lambda)$

#

Corollary [Properties of $PPP(\Lambda)$]

1. If $f \in L^1(\Lambda) \implies \mathbb{E}(N(f)) = \Lambda(f)$

$\longrightarrow$ for $f = \mathbb{1}_A$, we get $\mu(A) = \mathbb{E}(N(A)) = \Lambda(A)$ $\left(\begin{matrix}\text{mean} \\ = \\ \text{intensity}\end{matrix}\right)$

2. If $f \in L^2(\Lambda) \cap \underline{L^1(\Lambda)}_{*} \implies Var(N(f)) = \Lambda(f^2)$

$\longrightarrow$ for $f = \mathbb{1}_A$, we get $Var(N(A)) = \Lambda(A)$

* not obvious if Λ is not a finite measure

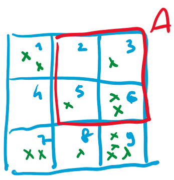
Verify! The proof is an application of the previous Theorem.

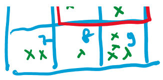
Example

1. Particles in boxes:

- Let E countable, $\mathcal{E} = \mathcal{P}(E)$, and $\Lambda \in \mathcal{M}(E)$
  $\hookrightarrow$ collection of boxes of $\mathbb{R}^d$
- Consider independent r.v. $(W_x)_{x \in E}$ s.t. $W_x \sim Poi(\Lambda(\{x\}))$
  $\hookrightarrow$ number of particles in the box x

Then $N(A) := \sum_{x \in E} W_x \underbrace{\mathbb{1}_A(x)}_{= \delta_x(A)}$, $\forall A \in \mathcal{E}$

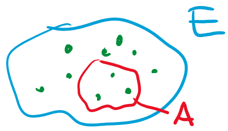


is a $PPP(\Lambda)$ describing the particles in boxes is a $PPP(\Lambda)$ describing the particles in boxes ↗ 

2. Stones in a field:

- Let $Y \sim Poi(\lambda)$ → number of stones
- Let $(X_n)_{n\in\mathbb{N}}$ iid on $(E, \mathcal{E})$, with law ν, independent of Y

↳ each stone is thrown, and takes a position in E, independently of the other, described by X_n (for stone n)



Then $$N(A) = \sum_{n=1}^{Y} \mathbb{1}_A(X_n), \quad \forall A \in \mathcal{E}$$

is a $PPP(\lambda \cdot \nu)$ describing the configuration of the thrown stones in E. Indeed, this corresponds to the example 1 above where we obtained (with updated notation)

$$\mathcal{L}_N(f) = e^{-\lambda(1-\nu(e^{-f}))} = e^{-\lambda\nu(1-e^{-f})}$$

↳ verify! → $\lambda \in \mathbb{R}^+$, $\nu \in \mathcal{M}(E)$