ADVANCED STOCHASTIC PROCESSES - 13° LECTURE

POISSON PROCESS

Def [Counting process]: A process $(N_t)_{t\geq 0}$ with value on $\mathbb{N}_0$ is a counting process if

$$N_0 = 0 \quad , \quad N_s \leq N_t \quad , \quad \forall\, s \leq t \quad (\text{non-decreasing})$$

Comment: N_t models the number of events of given type occurred (arrivals) in $[0,t]$ related to some random observation.

In modelizations there are two relevant properties that a counting process may satisfy:

1. (N_t) has independent increments if, $\forall\, 0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq t_m$, [ordered sequence of times in $\mathbb{R}^+$]
   $(N_{t_k} - N_{s_k})_{k=1,\dots,m}$ are independent r.v's
2. (N_t) has stationary increments if the law of an increment only depends on its length:
   $$N_{t+s} - N_s \stackrel{d}{=} N_t - N_0 = N_t \quad , \forall\, t,s \geq 0$$

Def [Poisson Process - homogeneous]

A counting process $(N_t)_{t\geq 0}$ is a (homogeneous) Poisson Process with intensity $\lambda > 0$ (denoted $PP(\lambda)$) if it has indep. and stat. increments s.t

3. $\forall\, s < t$: $N_t - N_s \sim Poi(\lambda \cdot (t-s))$

Remark: Note that $3. \iff 3'.$ $N_t \sim Poi(\lambda t)$ (from stationarity and $N_0 = 0$)

Q: Amplitude of jumps? How many arrivals at same time t?

On one hand $\mathbb{P}(N_h \geq 2) = 1 - e^{-h\lambda} - \lambda h e^{-h\lambda} \xrightarrow[h \to 0]{} 0$. (at most one jump in a infinitesimal interval of time)

On one hand $\mathbb{P}(N_h \geq 2) = 1 - e^{-h\lambda} - \lambda h e^{-h\lambda} \xrightarrow[h\to 0]{} 0$. (at most one jump in a infinitesimal interval of time)

But more than this:

The trajectories of $PP(\lambda)$ perform jumps of amplitude 1. In particular:

Proposition:

For any given $T \in \mathbb{R}^+$: $\mathbb{P}\left(\exists\, s \in [0,T] : \lim_{h \downarrow 0} (N_{s+h} - N_s) \geq 2\right) = 0$

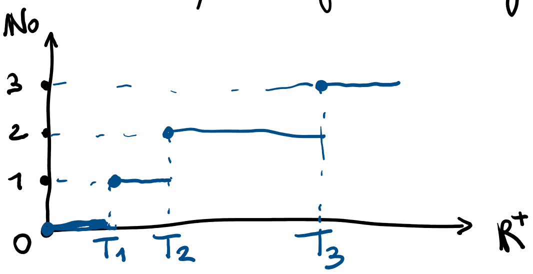
Proof: For $m \in \mathbb{N}$, consider a partition of $[0,T]$ in m intervals of length T/m : $\left(\left[k\cdot\frac{T}{m}\, ; (k+1)\cdot\frac{T}{m}\right]\right)_{k=0,\dots,m-1}$

Then, considering the complementary event:

$$\mathbb{P}\left(\bigcap_{k=0}^{m-1}\left\{\left(N_{\frac{k+1}{m}T} - N_{\frac{k}{m}T}\right) \leq 1\right\}\right) \underset{\text{indep, stat.}}{=} \mathbb{P}\left(N_{\frac{T}{m}} \leq 1\right)^m = e^{-T\cdot\lambda}\left(1 + \lambda\frac{T}{m}\right)^m$$

$$\xrightarrow[m\to\infty]{} e^{-T\lambda} \cdot e^{\lambda T} = 1 \qquad \#$$

If we draw the graph $t \to N_t$, we get a trajectory with jumps as the following:



- Arrival and inter-arrival times

For a $PP(\lambda)$, $(N_t)_{t\geq 0}$ let us consider the sequence of arrival times

$\times$ $(T_m)_{m\geq 1}$: $T_m := \inf\{t \geq 0 : N_t = m\} \in \overline{\mathbb{R}}_+$ (with $\inf\{\emptyset\} = +\infty$)

and notice that $\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = e^{-\lambda t} \Rightarrow T_1 \sim Exp(\lambda)$

Moreover, we consider the seq. of inter-arrival times $(W_m)_{m\geq 1}$ s.t

Moreover, we consider the seq. of inter-arrival times $(W_n)_{n\geq 1}$ s.t

* $W_n := T_n - T_{n-1}, \ \forall n \geq 2$ so that $T_n = \sum_{k=1}^{n} W_k \ \forall n \in \mathbb{N}$
  $W_1 = T_1$

Proposition: $(W_n)_{n\geq 1}$ is a sequence of iid $Exp(\lambda)$ r.v.'s.

<u>Proof</u>: For a given $n \in \mathbb{N}$, we first compute the joint prob. density of $(T_1, \dots, T_n)$, and then use that $(W_1, \dots, W_n) = F(T_1, \dots, T_n)$ to compute its density and derive the result.

$$F = \begin{pmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix}$$

- Consider n disjoint intervals of length $h > 0$,
  $[t_k, t_k + h]$ with $t_k + h < t_{k+1}$, $\forall k = 1, \dots, n$

Then:
$$\mathbb{P}\left(\bigcap \{T_k \in (t_k, t_k+h]\}\right) = \mathbb{P}\left(N_{t_1} = 0, N_{t_1+h} - N_{t_1} = 1, N_{t_2} - N_{t_1+h} = 0, \dots\right)$$

indep. of increments

$$= \underline{e^{-\lambda t_1}} \cdot \underline{\lambda h e^{-\lambda h}} \, \underline{e^{-\lambda(t_2 - (t_1+h))}} \cdot \underline{\lambda h e^{-\lambda h}} \cdot \dots \cdot \underbrace{\lambda h e^{-\lambda h}}_{n\text{-th arrival}}$$

$$= e^{-\lambda(t_n + h)} \cdot \lambda^n \cdot h^n$$

Then $$\frac{\mathbb{P}\left(\bigcap \{T_k \in (t_k, t_k+h]\}\right)}{h^n} \xrightarrow[h \to 0]{} \lambda^n e^{-\lambda t_n} = p_{T_1,\dots,T_n}(t_1, \dots, t_n)$$

density of $(T_1, \dots, T_n)$

- Since $(W_1, \dots, W_n) = F(T_1, \dots, T_n)$
  with $F^{-1} = \begin{pmatrix} 1 & & 0 \\ 1 & 1 & \\ \vdots & & \ddots \\ 1 & & 1 \end{pmatrix}$, we get that $(W_1, \dots, W_n)$ has density

$$p_{W_1,\dots,W_n}(s_1, \dots, s_n) = \lambda^n e^{-\lambda(s_1 + \dots + s_n)} = \prod_{k=1}^{n} \lambda e^{-\lambda s_k} \Rightarrow (W_n) \text{ iid}, Exp(\lambda)$$

#

Corollary: $T_n = \sum_{k=1}^{n} W_k \sim \Gamma(n, \lambda)$

It also holds a converse result of the above Proposition, which

It also holds a converse result of the above Proposition, which indeed provides an alternative construction of $(N_t)_{t\geq 0}$, and shows that $PP(\lambda)$ exists!

Construction: Let $(W_n)_{n\in\mathbb{N}}$ iid rv's $\sim Exp(\lambda)$

Define $T_n := \sum_{k=1}^{n} W_k$, $\forall n\in\mathbb{N}$.

Set $N_t := \#\{n\in\mathbb{N}_0 : T_n \leq t\}$

Theorem: The process $(N_t)_{t\geq 0}$ is a $PP(\lambda)$

Proof (idea): Use $\{N_t = k\} = \{T_k \leq t < T_{k+1}\}$, and the joint density of $(T_1,\dots,T_n)$ to show that, (see Klenke - section 5.5)

$$\mathbb{P}(N_{t_j} - N_{t_{j-1}} = k_j,\ \forall j=1,\dots,m) = \prod_{j=1}^{m}\left[e^{-\lambda(t_j-t_{j-1})}\cdot\frac{(\lambda(t_j-t_{j-1}))^{k_j}}{k_j!}\right]$$

$\forall t_1 < t_2 < \dots < t_m$, $\forall k_1,\dots,k_m\in\mathbb{N}_0$, $\forall m$

Decomposition and superposition of Poisson Process

- Superposition

Let $(N^k)_{k\in M}$ a countable family of independent Poisson processes s.t

$$N^k = (N^k_t)_{t\geq 0} \text{ is a } PP(\lambda_k)$$

Interpretation: Each process counts events of a given mark $k\in M$

Let $N_t := \sum_{k\in M} N^k_t$, $\forall t\geq 0$. If $\lambda = \sum_{k\in M}\lambda_k < \infty$

$\Longrightarrow$ $(N_t)_{t\geq 0}$ is $PP(\lambda)$

Indeed: - (N_t) is a counting process $\begin{cases} N_0 = 0 \\ N_t \nearrow \text{ since } N^k_t \nearrow\ \forall k\in M \end{cases}$

- indep. of increments follows from indep. of increments of $(N^k)_{k\in M}$
- for any $0 < s < t$:

indep. of N^k, $k\in M$

- for any $0 < s < t$:

$$N_t - N_s = \sum_{k\in M} (N_t^k - N_s^k) \overset{d}{=} \sum_{k\in M} Poi((t-s)\lambda_k) \overset{\text{indep. of } N^k,\, k\in M}{=} Poi((t-s)\lambda)$$

#

- Decomposition: Marked Point Process

Let $(N_t)_{t\geq 0}$ be a $PP(\lambda)$ and assume that each arrival event, indep. on the other, has a random mark in a countable set M with density $p=(p_k)_{k\in M} \in \mathcal{P}(M)$. → Also represented as: $(T_n, m_n)_{n\geq 1}$, m_n → mark of n-th arrival $\sim p$, T_n ↳ n-th arrival time $\sim \Gamma(n,\lambda)$

Let $N_t^k = \#$ arrivals of mark k in $[0,t]$ $\forall t\geq 0$ and $\forall k\in M$

$\Rightarrow$ $N^k = (N_t^k)$ is a $PP(\lambda p_k)$ and $(N^k)_{k\in M}$ are independent

Indeed: • N^k are all counting processes by construction

• N^k have all indep. increments, since N has indep. increments

Moreover, for any $0<s<t$, $\forall (n_k)_{k\in M}$ s.t. $\sum_{k\in M} n_k = n < \infty$

$$P(N_t^k - N_s^k = n_k, \forall k\in M) = \underbrace{P(N_t^k - N_s^k = n_k, \forall k \in M \mid N_t - N_s = n)}_{\text{combinatorial term}} \underbrace{P(N_t - N_s = n)}_{\frac{(\lambda(t-s))^n}{n!} e^{-\lambda(t-s)}}$$

$$= \frac{\cancel{n!}}{\prod_{k\in M} n_k!} \left(\prod_{k\in M} p_k^{n_k}\right) \frac{[\lambda(t-s)]^n}{\cancel{n!}} e^{-\lambda(t-s)}$$

$$= \prod_{k\in M} \left[\frac{(p_k \cdot \lambda(t-s))^{n_k}}{n_k!} \cdot e^{-\lambda p_k (t-s)} \right]$$

$n = \sum_{k\in M} n_k$, $1 = \sum_{k\in M} p_k$

which implies that (N^k) are independent, and all with stationary increments s.t. $N_t^k - N_s^k \sim Poi(\lambda \cdot p_k (t-s))$. #

Generalized Poisson Processes

- Inhomogeneous Poisson Processes

- ## Inhomogeneous Poisson Processes

We want remove the stationary hypothesis in the definition of $PP(\lambda)$, hence allowing inhomogeneity among increments of equal lenght.

Def: Let $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ continuous (rate) function.
An Inhomogeneous Poisson Process with rate function λ ($IPP(\lambda)$) is a counting process with independent increments and s.t.

(1) $N_b - N_a \sim Poi\left(\int_a^b \lambda(x)dx\right)$, $\forall\ 0 \leq a \leq b$

Notation: $\Lambda(a,b) = \int_a^b \lambda(x)dx$

so that $P(N_b - N_a = k) = \frac{(\Lambda(a,b))^k}{k!} e^{-\Lambda(a,b)}$

Remark:

- for $\lambda(t) = \lambda$ (constant rate function $\Rightarrow PP(\lambda)$) with $\Lambda[0,t] = \lambda t$
- (1) is also equivalent to $\begin{cases} \bullet\ P(N_{t+h} - N_t = 1) = h\lambda(t) + o(h) \\ \bullet\ P(N_{t+h} - N_t \geq 2) = o(h) \end{cases}$

Compound Poisson Process

Def: Let $N = (N_t)_{t \geq 0}$ be a $PP(\lambda)$ and $(Y_k)_{k\in\mathbb{N}}$ iid $\sim \nu \in \mathcal{P}(\mathbb{R})$ independent of N. The compound Poisson process $(X_t)_{t\geq 0}$ is given by

$$X_t = \sum_{k=1}^{N_t} Y_k \quad , \text{ with } X_t = 0 \text{ if } N_t = 0$$

Example:

- $N_t = \#$ accidents reported up to time t $\Rightarrow X_t =$ total damage up to time t.

- N_t = # accidents reported up to time t
- Y_k = amount of damage of the k-th accident

$\Rightarrow X_t$ = total damage up to time t

Remark:

- If $Y_k = 1$, $\forall k \in \mathbb{N} \Rightarrow PP(\lambda)$
- In general (X_t) performs big jumps, even negative.

Average of X_t: $\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t | N_t)) \underset{\uparrow}{=} \mathbb{E}(m \cdot N_t) = m \cdot \lambda t$

let $m := \mathbb{E}(Y_k)$

Variance of X_t: $Var(X_t) \underset{\uparrow}{=} \sigma^2 \, \mathbb{E}(N_t) + m^2 Var(N_t) = \sigma^2 \lambda t + m^2 \lambda t$

$= \underbrace{(\sigma^2 + m^2)}_{= \mathbb{E}(Y^2)} \lambda t$

let $\sigma^2 = Var(Y_k)$

Generating function of X_t (Laplace transform)

Notation: for a real r.v X, consider the generating functions

$$\mathcal{L}_X(s) := \mathbb{E}(e^{-sX}), \; s \in \mathbb{R}^+ \quad \text{and} \quad G_X(u) := \mathbb{E}(u^X), \; u \in [0,1]$$

$$\mathcal{L}_{X_t}(s) = \mathbb{E}(e^{-sX_t}) = \mathbb{E}\left(\mathbb{E}(e^{-sX_t} | N_t)\right) = \mathbb{E}\left(\prod_{k=1}^{N_t} \mathbb{E}(e^{-sY_k})\right)$$

$$= \mathbb{E}\left((\mathcal{L}_Y(s))^{N_t}\right) = G_{N_t}(\mathcal{L}_Y(s)) \overset{*}{=} e^{-\lambda t(1 - \mathcal{L}_Y(s))}$$

where $G_{N_t}(u) = \sum_{k=1}^{\infty} \frac{u^k}{k!} (\lambda t)^k e^{-\lambda t} = e^{-\lambda t} \cdot e^{\lambda t u} = e^{-\lambda t(1-u)}$