

ADVANCED STOCHASTIC PROCESSES - 13th LECTURE

POISSON PROCESS

Def [Counting process]: A process $(N_t)_{t \geq 0}$ with values on \mathbb{N}_0 is a counting process if

$$N_0 = 0, \quad N_s \leq N_t, \quad \forall s \leq t \quad (\text{non-decreasing})$$

Comment: N_t models the number of events of given type occurred in $[0, t]$ related to some random observations. (arrivals)

In modelisations there are two relevant properties that a counting process may satisfy:

1. (N_t) has independent increments if, $\forall 0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq t_m$,
($N_{t_k} - N_{s_k}$)_{k=1, \dots, m} are independent r.v.'s
ordered sequence of times in \mathbb{R}^+

2. (N_t) has stationary increments if the law of an increment only depends on its length:

$$N_{t+s} - N_s \stackrel{d}{=} N_t - N_0 = N_t, \quad \forall t, s \geq 0$$

Def [Poisson Process - homogeneous]

A counting process $(N_t)_{t \geq 0}$ is a (homogeneous) Poisson Process with intensity $\lambda > 0$ (denoted PP(λ)) if it has indep. and stat. increments s.t.

3. $\forall s < t: N_t - N_s \sim \text{Poi}(\lambda \cdot (t-s))$

↳ Remark: Note that 3. \Leftrightarrow 3'. $N_t \sim \text{Poi}(\lambda t)$ (from stationarity and $N_0 = 0$)

Q: Amplitude of jumps? How many arrivals at some time t ?

On one hand $P(N_h \geq 2) = 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} \xrightarrow{h \rightarrow 0} 0$. (at most one jump in an infinitesimal interval of time)

Moreover, we consider the seq. of inter-arrival times $(W_m)_{m \geq 1}$ s.t

$$* W_m := T_m - T_{m-1}, \forall m \geq 2 \quad \text{so that} \quad T_m = \sum_{k=1}^m W_k \quad \forall m \in \mathbb{N}$$

$$W_1 = T_1$$

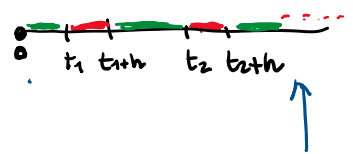
Proposition: $(W_m)_{m \geq 1}$ is a sequence of iid $\text{Exp}(\lambda)$ r.v.'s.

Proof: For a given $m \in \mathbb{N}$, we first compute the joint prob. density of (T_1, \dots, T_m) , and then use that $(W_1, \dots, W_m) = F(T_1, \dots, T_m)$ to compute its density and derive the result.

$$F = \begin{pmatrix} 1 & & 0 \\ -1 & 1 & \\ & \ddots & \ddots \\ 0 & & -1 & 1 \end{pmatrix}$$

• Consider m disjoint intervals of length $h > 0$,

$$(t_k, t_k+h] \quad \text{with} \quad t_{k+h} < t_{k+1}, \quad \forall k=1, \dots, m$$



Then:

$$\begin{aligned} P(\bigcap \{T_k \in (t_k, t_k+h]\}) &= P(N_{t_1}=0, N_{t_2+h}-N_{t_1}=1, N_{t_2}-N_{t_2+h}=0, \dots) \\ &\stackrel{\text{indep. of increments}}{=} \underbrace{e^{-\lambda t_1}} \cdot \underbrace{\lambda h e^{-\lambda h}} \cdot \underbrace{e^{-\lambda(t_2 - (t_1+h))}} \cdot \underbrace{\lambda h e^{-\lambda h}} \cdot \dots \cdot \underbrace{\lambda h e^{-\lambda h}}_{\rightarrow m\text{-th arrival}} \\ &= e^{-\lambda(t_m+h)} \cdot \lambda^m \cdot h^m \end{aligned}$$

$$\text{Then } \frac{P(\bigcap \{T_k \in (t_k, t_k+h]\})}{h^m} \xrightarrow{h \rightarrow 0} \lambda^m e^{-\lambda t_m} = P_{T_1, \dots, T_m}(t_1, \dots, t_m)$$

density of (T_1, \dots, T_m)

• Since $(W_1, \dots, W_m) = F(T_1, \dots, T_m)$

with $F^{-1} = \begin{pmatrix} 1 & & 0 \\ -1 & 1 & \\ & \ddots & \ddots \\ 0 & & -1 & 1 \end{pmatrix}$, we get that (W_1, \dots, W_m) has density

$$P_{W_1, \dots, W_m}(w_1, \dots, w_m) = \lambda^m e^{-\lambda(w_1 + \dots + w_m)} = \prod_{k=1}^m \lambda e^{-\lambda w_k} \Rightarrow (W_m) \text{ iid, } \text{Exp}(\lambda)$$

≠

Corollary: $T_m = \sum_{k=1}^m W_k \sim \Gamma(m, \lambda)$

It also holds a converse result of the above Proposition, which

It also holds a converse result of the above Proposition, which indeed provides an alternative construction of $(N_t)_{t \geq 0}$, and shows that $PP(\lambda)$ exists!

Construction: Let $(W_m)_{m \in \mathbb{N}}$ iid r.v.'s $\sim \text{Exp}(\lambda)$

Define $T_m := \sum_{k=1}^m W_k$, $\forall m \in \mathbb{N}$.

Set $N_t := \# \{m \in \mathbb{N}_0 : T_m \leq t\}$

Theorem: The process $(N_t)_{t \geq 0}$ is a $PP(\lambda)$

Proof (idea): Use $\{N_t = k\} = \{T_k \leq t < T_{k+1}\}$, and the joint density of (T_1, \dots, T_m) to show that, (see Klenke - section 5.5)

$$P(N_{t_j} - N_{t_{j-1}} = k_j, \forall j=1, \dots, m) = \prod_{j=1}^m \left[e^{-\lambda(t_j - t_{j-1})} \cdot \frac{(\lambda(t_j - t_{j-1}))^{k_j}}{k_j!} \right]$$

$\forall t_1 < t_2 < \dots < t_m, \forall k_1, \dots, k_m \in \mathbb{N}_0, \forall m$

Decomposition and superposition of Poisson Process

• Superposition

Let $(N^k)_{k \in M}$ a countable family of independent Poisson processes s.t

$$N^k = (N_t^k)_{t \geq 0} \text{ is a } PP(\lambda_k)$$

Interpretation: Each process counts events of a given mark $k \in M$

Let $N_t := \sum_{k \in M} N_t^k$, $\forall t \geq 0$. If $\lambda = \sum_{k \in M} \lambda_k < \infty$

$\Rightarrow (N_t)_{t \geq 0}$ is $PP(\lambda)$

Indeed: (N_t) is a counting process $\left\{ \begin{array}{l} N_0 = 0 \\ N_t \uparrow \text{ since } N_t^k \uparrow \forall k \in M \end{array} \right.$

• indep. of increments follows from indep. of increments of $(N^k)_{k \in M}$

• for any $0 < s < t$:

indep. of $N^k, k \in M$

• for any $0 < s < t$:

$$N_t - N_s = \sum_{K \in M} (N_t^K - N_s^K) \stackrel{!}{=} \sum_{K \in M} \text{Poi}((t-s)\lambda_K) \stackrel{!}{=} \text{Poi}((t-s)\lambda)$$

indep. of $N^K, K \in M$

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• Decomposition: Marked Point Process

Let $(N_t)_{t \geq 0}$ be a $PP(\lambda)$ and assume that each arrival event, indep. on the other, has a random mark in a countable set M with

density $P = (P_K)_{K \in M} \in \mathcal{P}(M)$.

Also represented as:

$(T_m, m_m)_{m \geq 1}$ → mark of m -th arrival $\sim P$
 $\hookrightarrow m$ -th arrival time $\sim \Gamma(m, \lambda)$

Let $N_t^K = \#$ arrivals of mark K in $[0, t]$

$\forall t \geq 0$ and $\forall K \in M$

$\Rightarrow \underline{N^K} = (N_t^K)_{t \geq 0}$ is a $PP(\lambda P_K)$ and $\underline{(N^K)}_{K \in M}$ are independent

Indeed: • N^K are all counting processes by construction

• N^K have all indep. increments, since N has indep. increments

Moreover, for any $0 < s < t$, $\forall (m_K)_{K \in M}$ s.t. $\sum_{K \in M} m_K = m < \infty$

$$P(N_t^K - N_s^K = m_K, \forall K \in M) = \underbrace{P(N_t^K - N_s^K = m_K, \forall K \in M | N_t - N_s = m)}_{\text{combinatorial term}} \underbrace{P(N_t - N_s = m)}_{\frac{(\lambda(t-s))^m e^{-\lambda(t-s)}}{m!}}$$

$$= \frac{m!}{\prod_{K \in M} m_K!} \left(\prod_{K \in M} P_K^{m_K} \right) \frac{[\lambda(t-s)]^m e^{-\lambda(t-s)}}{m!}$$

$$= \prod_{K \in M} \left[\frac{(P_K \cdot \lambda(t-s))^{m_K}}{m_K!} \cdot e^{-\lambda P_K(t-s)} \right]$$

$m = \sum_{K \in M} m_K$
 $1 = \sum_{K \in M} P_K$

which implies that (N^K) are independent, and all with stationary increments s.t. $N_t^K - N_s^K \sim \text{Poi}(\lambda \cdot P_K(t-s))$. #

Generalized Poisson Processes

• Inhomogeneous Poisson Processes

• Inhomogeneous Poisson Processes

We want remove the stationary hypothesis in the definition of PP(λ), hence allowing inhomogeneity among increments of equal length.

Def: Let $\lambda: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous (rate) function.

An Inhomogeneous Poisson Process with rate function λ (IPP(λ)) is a counting process with independent increments and s.t.

$$(1) \quad N_b - N_a \sim \text{Poi}(\Lambda(a,b)), \quad \forall 0 \leq a < b$$

$$\text{with} \quad \Lambda(a,b) = \int_a^b \lambda(x) dx$$

$$\hookrightarrow \forall k \in \mathbb{N}_0: P(N_b - N_a = k) = \frac{(\Lambda(a,b))^k}{k!} e^{-\Lambda(a,b)}$$

Remark:

• for $\lambda(t) = \lambda$ (constant rate function) \Rightarrow PP(λ)

$$\text{with} \quad \Lambda[0,t] = \lambda t$$

Compound Poisson Process

Def: Let $N = (N_t)_{t \geq 0}$ be a PP(λ) and $(Y_k)_{k \in \mathbb{N}}$ iid $\sim Y \in \mathcal{B}(\mathbb{R})$ independent of N . The compound Poisson process $(X_t)_{t \geq 0}$ is given by

$$X_t = \sum_{k=1}^{N_t} Y_k, \quad \text{with } X_t = 0 \text{ if } N_t = 0$$

Example:

• $N_t = \#$ accidents reported up to time t

$\Rightarrow X_t =$ total damage up to time t

- $N_t = \#$ accidents reported up to time $t \Rightarrow X_t =$ total damage up to time t
- $Y_k =$ amount of damage of the k -th accident

Remark:

- If $Y_k = 1, \forall k \in \mathbb{N} \Rightarrow PP(\lambda)$
- In general (X_t) performs big jumps, even negative.

Average of X_t : $E(X_t) = E(E(X_t | N_t)) = E(m \cdot N_t) = m \cdot \lambda t$
 let $m := E(Y_k)$

Variance of X_t : $Var(X_t) = \sigma^2 E(N_t) + m^2 Var(N_t) = \sigma^2 \lambda t + m^2 \lambda t = (\sigma^2 + m^2) \lambda t = E(Y^2) \lambda t$
 let $\sigma^2 = Var(Y_k)$

Generating function of X_t (Laplace transform)

Notation: for a real r.v. X , consider the generating functions $L_X(\theta) := E(e^{-\theta X}), \theta \in \mathbb{R}^+$ and $G_X(u) := E(u^X), u \in [0, 1]$

$$L_{X_t}(\theta) = E(e^{-\theta X_t}) = E(E(e^{-\theta X_t} | N_t)) = E\left(\prod_{k=1}^{N_t} E(e^{-\theta Y_k})\right) \\ = E((L_Y(\theta))^{N_t}) \stackrel{*}{=} G_{N_t}(L_Y(\theta)) \stackrel{*}{=} e^{-\lambda t(1 - L_Y(\theta))}$$

$\in [0, 1]$

* where, for $u \in [0, 1]$,

$G_{N_t}(u) := E(u^{N_t})$ "probability generating function"
 $= \sum_{k=0}^{\infty} \frac{u^k}{k!} (\lambda t)^k \cdot e^{-\lambda t} = e^{-\lambda t} \cdot e^{-\lambda t u} = e^{-\lambda t(1-u)}$