

LCD (30/03)

Bisimilarity as a Fixpoint

bisimulation \rightsquigarrow largest \sim



alternative: fixpoint

→ theoretically interesting

→ provides an algorithm

→ assertion language

recursion

$\text{fact} : \mathbb{N} \rightarrow \mathbb{N}$

$$\text{fact}(m) = \begin{cases} 1 & \text{if } m=0 \\ m \times \text{fact}(m-1) & \text{if } m>0 \end{cases}$$

we are dealing with functions

$$\mathcal{F} = \{ f \mid f: \mathbb{N} \rightarrow \mathbb{N} \}$$

$F: \mathcal{F} \rightarrow \mathcal{F}$

$$f \mapsto F(f) \quad \text{with} \quad F(f)(m) = \begin{cases} 1 & \text{if } m=0 \\ m \times f(m-1) & \text{if } m>0 \end{cases}$$

we define the factorial $\text{fact} \in \mathcal{F}$ such that

$$F(\text{fact}) = \text{fact}$$

i.e. fact is a fixpoint of F.

Example:

define $x_0 \in \mathbb{R}$ s.t. it satisfies

$$x_0 = \frac{\sqrt{x_0} + e^{x_0}}{\log x_0} \quad ??$$

KNASTER-TARSKI'S THEOREM

fixed point for monotone functions over complete lattices

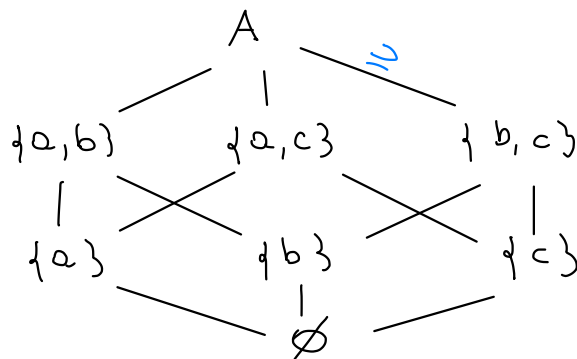
SPECIAL CASE: powerset lattice

A set

$$2^A = \{x \mid x \subseteq A\}$$

ordered by \subseteq

if $A = \{a, b, c\}$



* Monotone function

$f: 2^A \rightarrow 2^A$ is monotone if $\forall x, y \in 2^A$

$x \subseteq y$ then $f(x) \subseteq f(y)$

example:

$$f: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$$

$$f(x) = \{0\} \cup \{2+x \mid x \in x\}$$

(fixpoints? \mathbb{P} $f(\mathbb{N}) = \mathbb{N} \setminus \{1\}$)

$$f(x) = \{0\} \cup x \cup \{2+x \mid x \in \mathbb{N}\}$$

(fixpoints: $\mathbb{P}, \mathbb{N}, \dots$)

$$g(x) = \{0\} \cup \{2+x \mid x \notin x\}$$

NON
MONOTONE

$$g(\emptyset) = \mathbb{N} \setminus \{1\} \neq \{0\} = g(\mathbb{N}) \quad \text{but } \emptyset \subseteq \mathbb{N}$$

Knaster-Tarski's Theorem

Given $f: Z^A \rightarrow Z^A$ monotone, then f has

(1) greatest fixpoint $\text{gfp}(f) = \bigcup \{x \in Z^A \mid x \leq f(x)\}$

(2) least fixpoint $\text{lfp}(f) = \bigcap \{x \in Z^A \mid f(x) \leq x\}$

proof

we prove (1). Let $X_M = \bigcup \underbrace{\{x \in Z^A \mid x \leq f(x)\}}_{\text{Post}}$

(a) X_M is a fixpoint $f(X_M) = X_M$

(b) X_M is the greatest fixpoint $\forall x \in Z^A \quad f(x) = x \Rightarrow x \leq X_M$

(a) we start with (i) $X_M \in \text{Post}$ $X_M \leq f(X_M)$ (*)

for each $x \in \text{Post}$ $x \leq \bigcup \text{Post} = X_M$

then $x \leq f(x) \leq f(X_M)$
 \uparrow $x \in \text{Post}$ \uparrow f monotone

Thus $X_M = \bigcup \text{Post} \leq f(X_M)$

(ii) $f(X_M) \in \text{Post}$

from (*) and monotonicity

$$f(X_M) \leq f(f(X_M))$$

thus

$X_M = \bigcup \text{Post} \geq f(X_M)$

from (i) and (ii)

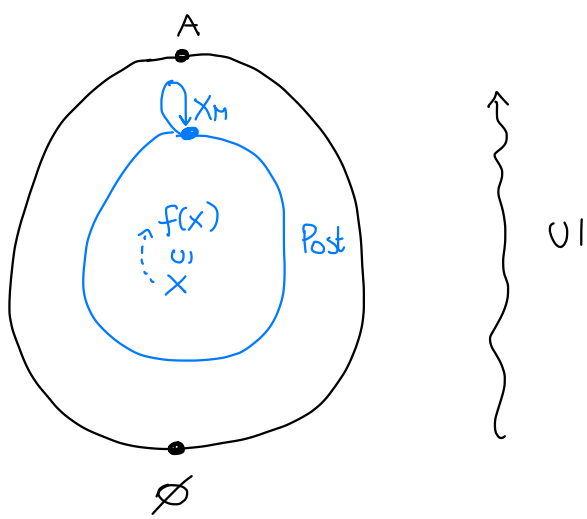
$X_M = f(X_M)$

(b) X_M is the largest fixpoint: let $x \in Z^A$ s.t. $x = f(x)$

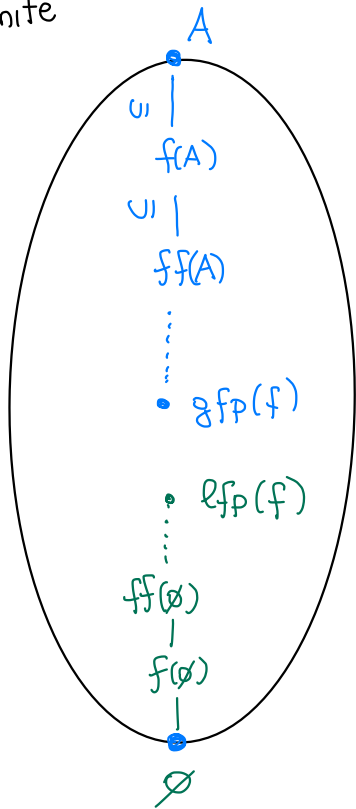
hence $x \in \text{Post}$. Thus $x \leq \bigcup \text{Post} = X_M$.

□

2^A



* If A is finite



OBSERVATION : If A is finite and $f: 2^A \rightarrow 2^A$ monotone

(a) $lfp(f) = f^k(\emptyset)$ for some $k \leq |A|$

(b) $gfp(f) = f^h(A)$ " " $h \leq |A|$

where $f^k(x) = \underbrace{f \dots f}_{k \text{ times}}(x)$

* Bisimilarity as a fixpoint

a bisimulation is a relation $R \subseteq \text{Proc} \times \text{Proc}$ s.t.

if $P R Q$ then (i) if $P \xrightarrow{\alpha} P'$ then $Q \xrightarrow{\alpha} Q'$ and $P' R Q'$
 (ii) if $Q \xrightarrow{\alpha} Q'$ then $P \xrightarrow{\alpha} P'$ and $P' R Q'$
 $(P, Q) \in R$ $(P', Q') \in R$

we can consider

$$F: 2^{\text{Proc} \times \text{Proc}} \rightarrow 2^{\text{Proc} \times \text{Proc}}$$

$$F(R) = \left\{ (P, Q) \mid \begin{array}{l} \text{(i) if } P \xrightarrow{\alpha} P' \text{ then } Q \xrightarrow{\alpha} Q' \text{ and } (P', Q') \in R \\ \text{(ii) if } Q \xrightarrow{\alpha} Q' \text{ then } P \xrightarrow{\alpha} P' \text{ and } (P', Q') \in R \end{array} \right\}$$

some observations

(1) Given $R \in 2^{\text{Proc} \times \text{Proc}}$ R is a bisimulation if

if $(P, Q) \in R$ then $(P, Q) \in F(R)$

i.e. $R \subseteq F(R)$ R is a post fixpoint of F

(2) F monotone

(3) we defined

$$\mathcal{N} = \bigcup \{ R \mid R \text{ bisimulation} \} = \bigcup \{ R \mid R \subseteq F(R) \} = \text{gfp}(F)$$

(even 0 is even
if x even $\Rightarrow 2+x$ even)

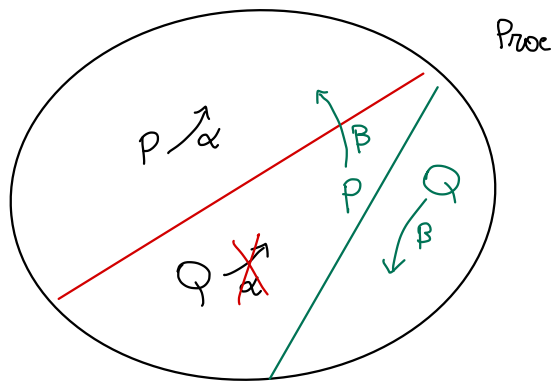
* If processes are finite state $F: 2^{\text{Proc} \times \text{Proc}} \rightarrow 2^{\text{Proc} \times \text{Proc}}$

$$\mathcal{N} = \text{gfp}(F) = F^k(\text{Proc} \times \text{Proc})$$

$$\text{Proc} \times \text{Proc} \supseteq F(\text{Proc} \times \text{Proc}) \supseteq F F(\text{Proc} \times \text{Proc}) \supseteq \dots$$

$$\dots \supseteq F^k(\text{Proc} \times \text{Proc}) = F^{k+1}(\text{Proc} \times \text{Proc}) = \mathcal{N}$$

↑
equivalences



$$F(R) = \left\{ (P, Q) \mid \begin{array}{l} \text{(i) if } P \xrightarrow{\alpha} P' \text{ then } Q \xrightarrow{\alpha} Q' \text{ and } (P', Q') \in R \\ \text{(ii) if } Q \xrightarrow{\alpha} Q' \text{ then } P \xrightarrow{\alpha} P' \text{ and } (P', Q) \in R \end{array} \right\}$$

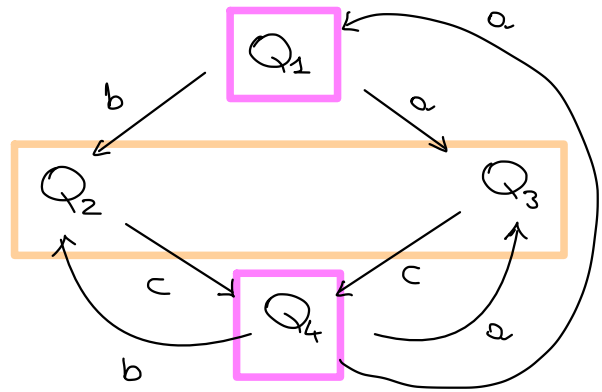
Example :

$$Q_1 = b \cdot Q_2 + a \cdot Q_3$$

$$Q_2 = c \cdot Q_4$$

$$Q_3 = c \cdot Q_4$$

$$Q_4 = b \cdot Q_2 + a \cdot Q_3 + a \cdot Q_1$$



$$Proc = \{Q_1, Q_2, Q_3, Q_4\}$$

$$F^0(Proc \times Proc) = Proc \times Proc \quad I = \{(P, P) \mid P \in Proc\}$$

$$F(Proc \times Proc) = \{(Q_1, Q_4), (Q_4, Q_1), (Q_2, Q_3), (Q_3, Q_2)\} \cup I$$

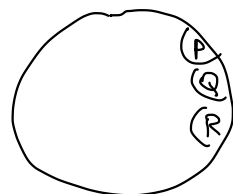
$$F^2(Proc \times Proc) = \{(Q_2, Q_3), (Q_3, Q_2)\} \cup I$$

$$F^3(Proc \times Proc) = \{(Q_2, Q_3), (Q_3, Q_2)\} \cup I = \infty$$

* Complexity (naive)

$m = \# \text{ states}$

$m = \# \text{ transitions}$



number of iterations ?

m

cost per iteration

m^2

} $O(m^2 m)$

You can do better

$O(m^2)$

Kamnelovskiy - Smolka '83

$O(m \log m)$

Page - Torzhan '87

↳ data structures for partitions

Trace equivalence

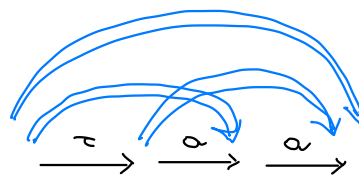
P-space complete

NP

EXP

Weak Bisimilarity

saturation by weak transitions



(quadratic increase)

What about infinite states

CCS bisimilarity is undecidable

• BPP (basic parallel processes) : CCS without synchronisation

→ bisimilarity is decidable

→ trace equivalence is undecidable

⋮

* Petri Nets

all equivalences are undecidable reachability is decidable

[not primitive recursive]