ADVANCED STOCHASTIC PROCESSES - 11° LECTURE

Branching processes

They are MC's which model the genealogical evolution of a population generated from one individual.

- let $p = (p_k)_{k \in \mathbb{N}_0} \in \mathcal{P}(\mathbb{N}_0)$ (offspring distribution)
- Consider i.i.d random variables $(X_{n,j})_{\substack{n \in \mathbb{N}_0 \\ j \in \mathbb{N}_0}}$ with density p

  $(\mathbb{P}(X_{n,j} = k) = p_k)$

  and define recursively the integer r.v.'s $(Z_n)_{n \in \mathbb{N}_0}$ s.t.

$$Z_0 = 1 \quad , \quad Z_{n+1} = \sum_{j=0}^{Z_n} X_{n,j}$$

Def: $(Z_n)_{n \geq 0}$ is called Galton-Watson process or branching process with offspring distribution p.

Interpretation:

- $X_{n,j} = \#$ offspring of the j-th individual in the n-th generation
- $Z_n = \#$ individual in the n-th generation

We then have a graphical representation as random tree:

where we put an edge among any individual and its progeny

Note that the corresponding tree can be finite or infinite, also depending on the choice of p.

Note also that the i.i.d. assumption over $(X_{n,j})$ corresponds to

Note also that the i.i.d. assumption over $(X_{n,j})$ corresponds to the hypothesis that each individual, independently of the others, generate an offspring with size distribution p.

By definition, $(Z_n)_{n\geq 0}$ is a MC on $\mathbb{N}_0$ with initial measure δ_1 and transition probabilities:

"$p^{*j}(i)$" convolution of p

$$\mathbb{P}(Z_{n+1}=k \mid Z_n=j) = \mathbb{P}\Big(\sum_{l=1}^{j} X_{n,l}=k\Big) = \sum_{\substack{l_1,\dots,l_j:\\ l_1+\dots+l_j=k}} p_{l_1}\cdots p_{l_j}$$

In particular (for $p_0\neq 0$) the Markov chain has the absorbing state 0 (the population dies out), and a relevant question is about survival or extinction.

Notation: Set

$$\eta := \mathbb{P}(\exists\, n : Z_n = 0) = \mathbb{P}\Big(\bigcup_m \{Z_m=0\}\Big)$$ extinction probability

$$\zeta := 1-\eta = \mathbb{P}(Z_n>0,\ \forall n\in\mathbb{N}) = \mathbb{P}\Big(\bigcap_m \{Z_m>0\}\Big)$$ survival probability

$$\mu := \sum_{k=1}^{\infty} k\cdot p_k \equiv \mathbb{E}(X_{n,j}) = \mathbb{E}(Z_1) \quad \forall n,j\in\mathbb{N}_0$$ average offspring size

Theorem [extinction of GW-process]

If p is non-trivial $(p_0+p_1\neq 1)$

absorbtion

$$\begin{cases} * \text{ if } \mu\leq 1 \implies \eta = 1 & \text{(a.s. extinction)}\\ * \text{ if } \mu> 1 \implies \eta < 1 & (\zeta>0 \text{ positive survival probability})\end{cases}$$

Moreover η is the smallest solution of the fixed point equation

$$s = G(s) \qquad \text{where} \qquad G(s) := \sum_{k\geq 0} s^k p_k$$ generating function of $X_{n,j}$ (and Z_1)

Proof: First note that $\{Z_n=0\} \subseteq \{Z_{n+1}=0\}, \forall n \in \mathbb{N}$, (increasing events)

Hence $\underline{\eta = \mathbb{P}\left(\bigcup_n \{Z_n=0\}\right) = \lim_{n\to\infty} \mathbb{P}(Z_n=0)}$.

For $n\in\mathbb{N}$, let $G_n(s) := \mathbb{E}(s^{Z_n})$, $\forall s \in [0,1]$ (generating function of Z_n)

so that $G_n(0) = \mathbb{P}(Z_n=0)$, $\forall n \in \mathbb{N}$ and with $G_1 \equiv G$.

Properties of G:

1. **Lemma**: $G_n(s) = G_{n-1}(G(s)) = G(G_{n-1}(s))$, $\forall n \geq 2$

Proof (of Lemma):

$$G_n(s) = \mathbb{E}(s^{Z_n}) = \mathbb{E}\left(\mathbb{E}(s^{Z_n} \mid Z_{n-1})\right) = \sum_{k\geq 0} \mathbb{P}(Z_{n-1}=k)\, \underbrace{\mathbb{E}\left(s^{\overbrace{\sum_{j=1}^{k} X_{n,j}}^{\text{sum of iid rv}}}\right)}_{= G(s)^k}$$

$$= \sum_{k\geq 0} \mathbb{P}(Z_{n-1}=k)\, G(s)^k = G_{n-1}(G(s)) \quad \text{(1° equality)}$$

Iterating: $G_n(s) = \overbrace{G\circ\cdots\circ G}^{n \text{ times}}(s) = G \circ G_{n-1}(s)$ (2° equality) #

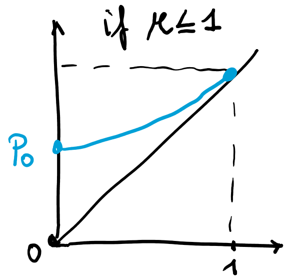
From the 2° id. of lemma, taken at $s=0$, we get $\mathbb{P}(Z_n=0) = G(\mathbb{P}(Z_{n-1}=0))$

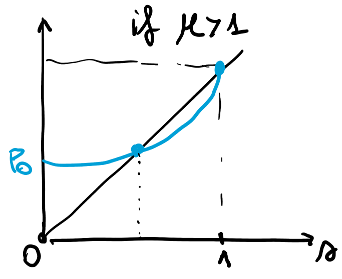
and taking the limit $n\to\infty$, $\boxed{\eta = G(\eta)}$

2. Graph of G for $s\in[0,1]$:

- $G(0) = \mathbb{P}(Z_1=0) = p_0$, $G(1) = 1$
- G is strictly convex in $(0,1)$: $\begin{cases} * \; G'(s) = \sum_{k\geq 1} k p_k s^{k-1} > 0 \\ * \; G''(s) = \sum_{k\geq 2} k(k-1) p_k s^{k-2} > 0 \end{cases}$
- $\lim_{s\uparrow 1} G'(s) = \mu$

Altogether:





0 —— 1 s 0 1 10

Then:

- if $\mu \le 1$, $\eta = G(\eta) \iff \boxed{\eta = 1.}$
- if $\mu > 1$, we prove by induction that a solution $r = G(r)$ is s.t. $\mathbb{P}(Z_n = 0) \le r$. Indeed

  G is $\nearrow$

  $* \ \mathbb{P}(Z_1 = 0) = G(0) \le G(r) = r$

  $* \ \mathbb{P}(Z_{n+1} = 0) \underset{\text{Lemma}}{=} G(\mathbb{P}(Z_n = 0)) \underset{\text{inductive hyp.}}{\le} G(r) = r$

Taking the limit $n \to \infty$, we obtain $\eta \le r \implies \eta$ is the smallest solution of $r = G(r)$, and hence $\eta < 1$. #

Branching Processes - asymptotic behavior

Let $T := \sum_{n=0}^{\infty} Z_n$ (size of the population, or of the GW tree)

and notice that if $\mu > 1$, then $\mathbb{P}(T = \infty) = \mathcal{G} = 1 - \eta > 0$

Average size:

$$\mathbb{E}(Z_n) = \mathbb{E}(\mathbb{E}(Z_n \mid Z_{n-1})) = \mathbb{E}(Z_{n-1}) \cdot \mu \overset{\text{iterating}}{=} \cdots = \mu^n$$

$\sum_{j=1}^{Z_{n-1}} X_{n-1,j}$

Then:

(1) $$\mathbb{E}(T) = \sum_{n \ge 0} \mathbb{E}(Z_n) = \sum_{n \ge 0} \mu^n = \begin{cases} \frac{1}{1-\mu} & \text{if } \mu < 1 \\ +\infty & \text{if } \mu \ge 1 \end{cases}$$

Remark: if $\mu = 1$, then the population dies out a.s, but it has infinite average size.

(2) $\mathbb{P}(Z_n > 0) \le \mu^n$ (by Markov inequality)

which is useful for $\mu < 1$, as it provides an exponential decay of the size.

A: Can we estimate $\mathbb{P}(Z_n > 0)$ when $\mu > 1$?

B: Can we say something about the distribution of T?

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B : Can we say something about the distribution of T?

For example, estimates of $\mathbb{P}(T > m)$ or $\mathbb{P}(T < m)$?

- For what concerns A, we prove the following result.

Let $M_n := \frac{Z_n}{\mu^n}$, $\forall n \in \mathbb{N}$, and note that $(M_n)_{n\geq 0}$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(Z_k, k \leq n)$: $M_n \in L^1$ with $\mathbb{E}(M_n) = 1$, and

$$\mathbb{E}(M_n | \mathcal{F}_{n-1}) = \frac{\mathbb{E}(Z_n | Z_{n-1})}{\mu^n} = \frac{\mu \cdot Z_{n-1}}{\mu^n} = M_{n-1}.$$

Theorem [Convergence of "rescaled generation size"]

(1) If $\mu > 1$, then $M_n \xrightarrow[n\to\infty]{a.s.} M_\infty$, with $\mathbb{E}(M_\infty) < \infty$

(2) Moreover, if the offspring distr. p is s.t. $Var(Z_1) = \sigma^2 < \infty$, then

$$\mathbb{E}(M_\infty) = 1$$

Proof: The first result follows from the martingale convergence theorem. Indeed $\mathbb{E}(|M_n|) = \mathbb{E}(M_n) = 1$ $\forall n \in \mathbb{N}$, hence the theorem applies and provides (1).

The second statement follows from the L^2-martingale convergence theorem if we prove that $\mathbb{E}(M_n^2) < C$, $\forall n$ and some finite C.

This equivalent to show that the variance is uniformly bounded.

Recall: Given iid (X_k), independent from a integer r.v. T, and setting $S_T = \sum_{k=1}^{T} X_k$, then

$$Var(S_T) = Var(T) \cdot \mathbb{E}(X_k)^2 + \mathbb{E}(T) \cdot Var(X_k) \quad (\text{Check})$$

Since $M_n = 1 \sum^{Z_{n-1}} X_{n,j}$, we get from the above identity

Since $M_n = \frac{1}{\mu^n}\sum_{j=1}^{Z_{n-1}} X_{n,j}$, we get from the above identity

$$Var(M_n) = \frac{1}{\mu^{2n}}\left(\mu^2 \cdot Var(Z_{n-1}) + \mu^{n-1}\cdot\sigma^2\right) = Var(M_{n-1}) + \frac{\sigma^2}{\mu^{n+1}}$$

$$\text{(iterating)} = \sigma^2 \sum_{k=2}^{n+1} \frac{1}{\mu^k} \underset{\text{if } \mu>1}{<} \sigma^2 \frac{\mu}{\mu-1} < \infty .$$

Since $(M_n)_{n\geq 0}$ is a unif. bounded in L^2 , then

$$M_n \xrightarrow[n\to\infty]{} M_\infty \quad \text{a.s.}, \; L^2 \text{ (and } L^1)$$

$$\Rightarrow \quad \mathbb{E}(M_\infty) = \lim_{n\to\infty} \mathbb{E}(M_n) = 1 . \qquad \#$$

Remark: the result (2), which holds under stronger hypotheses, garantee that M_∞ is not a trivial r.v. $(P(M_\infty > 0) > 0)$.
In this case, we may approximate $Z_n \approx M_\infty \mu^n$ for $n \gg 1$
$\longrightarrow$ μ^n is the correct scaling for Z_n.

- For what concerns B, we need two ingredients:
  1. the exploration process
  2. Chernoff bounds for sum of iid random variables

1. Exploration process

We explore the Galton-Watson tree through an iterative procedure that create a bijection among the tree and a RW with iid steps $(X_k - 1)$ where $X_k \overset{d}{=} X_{n,j}$ $\left(\text{distr. } P\right)$

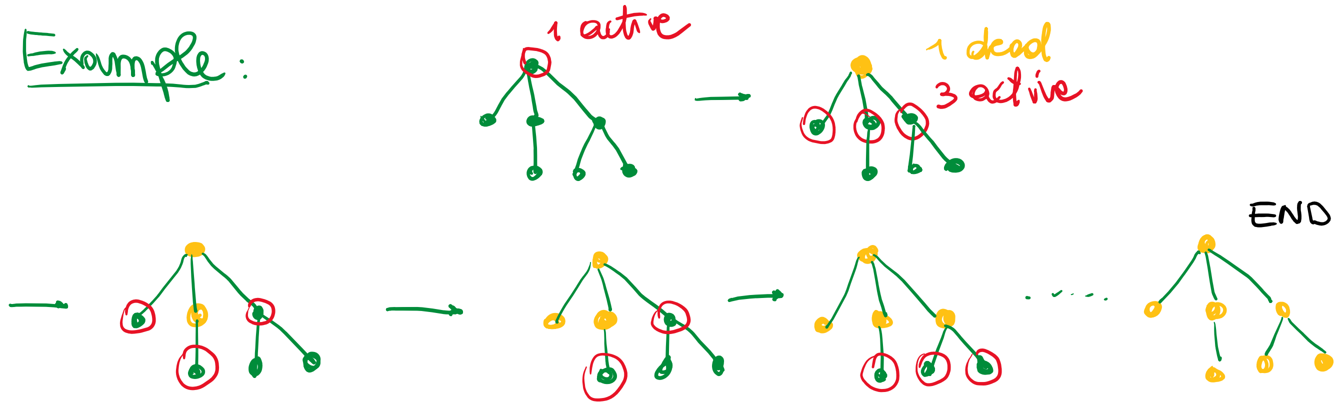
Def [exploration process]

- The process starts with 1 active vertex (e.g., the root)

- The process starts with 1 active vertex (= root)
- At each step, choose 1 vertex among the active ones (the order of the choice is uninfluent and can be deterministic):
  - the chosen vertex become dead
  - its neighbors become active
- Proceed iteratively until all the vertices of the GW-tree have been visited (activated) and then become dead.

  (possibly an infinite procedure if the tree is infinite)

Example:



Def: [RW associated with exploration process]

Let $(S_n)_{n\geq 0}$ defined setting

S_n = # active vertices in n iterations of the process

By construction it holds that:

$$S_0 = 1 \quad , \quad S_n = 1 + \sum_{k=1}^{n}(X_k - 1) = \overbrace{\sum_{k=1}^{n} X_k}^{=Y_n} - (n-1) = Y_n - (n-1)$$

where $(X_k)_{k\geq 1}$ are i.i.d. with distr. p (equiv. $X_k \overset{d}{=} X_{n,j}$)

Let $\tilde{T} = \inf\{n \geq 0 : S_n = 0\}$.

By construction: $T \overset{d}{=} \tilde{T}$

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size of population

hitting time in 0

Then, for any $\ell \in \mathbb{N}$:

$$\mathbb{P}(T > \ell) = \mathbb{P}(\tilde{T} > \ell) = \mathbb{P}(S_\ell > 0) = \mathbb{P}(Y_\ell \geq \ell - 1)$$

To estimate the last probability we use Chernoff bounds.