ADVANCED STOCHASTIC PROCESSES - 10° LECTURE

Harmonic functions w.r.t. P

- Let S be a countable set and X a MC on S with transition matrix P.

Def: A function $f: S \to \mathbb{R}$ is harmonic w.r.t. P if

$$Pf = f \quad \text{and hence} \quad (P - Id) f = 0$$

- It is subharmonic if $Pf \geq f$
- It is superharmonic if $Pf \leq f$

Equivalently: $\sum_{y \in S} P_{xy} \left(f(y) - f(x) \right) \overset{(\geq \text{ or } \leq)}{=} 0$ (where we used $\sum_{y \in S} P_{xy} f(x) = f(x)$)

Lemma: If f is bounded and (super-, sub-) harmonic, then $(f(X_n))_{n \geq 0}$ is a (super-, sub-) martingale w.r.t. $(\mathcal{F}_n)_{n \geq 0}$, the natural filtration.

Proof: $\mathbb{E}_x \left(f(X_n) \mid \mathcal{F}_{n-1} \right) \overset{a.s.}{=} \mathbb{E}_{X_{n-1}} \left(f(X_1) \right) = Pf(X_{n-1}) \overset{(\leq)}{=} f(X_{n-1})$

↳ by Markov property

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Remark:

If P corresponds to a SSRW on $\mathbb{Z}^d$, namely $P_{x,y} = \frac{1}{2d} \cdot \mathbb{1}_{\{|x-y|=1\}}$,

Then:

$$\begin{aligned}
(P - Id) f(x) &= \frac{1}{2d} \sum_{y : |y-x|=1} \left(f(y) - f(x) \right) \\
&= \frac{1}{2d} \underbrace{\sum_{k=1}^{d} \big(\overbrace{f(x+e_k) + f(x-e_k) - 2f(x)}^{= \text{ discrete } 2^\circ\text{-partial derivative w.r.t } e_k} \big)}_{\text{" discrete Laplacian}} \\
&= \frac{1}{2d} \Delta f(x)
\end{aligned}$$

In general, the operator $L := P - Id$ can be considered as a

$\frac{1}{2d}$

In general the operator $L := P - Id$ can be considered as a discrete differential operator acting on function $f : S \to \mathbb{R}$, and it is called generator. It's role will be fundamental when considering (continuous time) Markov Processes.

Dirichlet problem (on countable space S)

Let $D \subset S$, P a transition matrix on S and $\varphi : D^c \to \mathbb{R}$ a given "boundary condition". We look for function $f : S \to \mathbb{R}$ s.t

$D^c = S \setminus D$

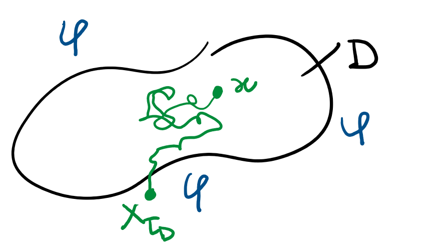
Dirichlet Problem (DP)

$$\begin{cases} Pf(x) = f(x) & \forall x \in D \\ f(x) = \varphi(x) & \forall x \in D^c \end{cases} \iff \begin{cases} Lf(x) = 0 & \forall x \in D \\ f(x) = \varphi(x) & \forall x \in D^c \end{cases}$$

The solution can be given in term of MC.

Theorem: Let $\tau_D = \inf\{n \geq 0 : X_n \in D^c\}$ (hitting time in D^c or exit time from D) and assume $\varphi : D^c \to \mathbb{R}$ is bounded. Then a solution of (DP) is (existence)

$$(1) \qquad f(x) := \mathbb{E}_x\left[\varphi(X_{\tau_D}) \mathbb{1}_{\{\tau_D < \infty\}}\right]$$



Proof:

- If $x \in D^c$, then $\tau_D = 0$ and
  $f(x) = \mathbb{E}_x(\varphi(X_{\tau_D})) = \varphi(x)$ ✓
- If $x \in D$, then the dynamics evolves at least of one step to reach D^c. Then, from the definition of the sol. $f(x)$ and conditioning on X_1, by the Markov property, we get:

and conditioning on X_1, by the Markov property, we get:

$$f(x) = \sum_{m=1}^{\infty} \mathbb{E}_x\left(\varphi(X_m)\mathbb{1}_{\{\tau_D=m\}}\right)$$

$x \notin D^c$, hence we can take off $m=0$

$$= \sum_{m=1}^{\infty}\sum_{y\in S} \mathbb{E}_x\left(\varphi(X_m)\mathbb{1}_{\{\tau_D=m\}} \mid X_1=y\right)\cdot P_{x,y}$$

$$= \sum_{y\in S} P_{x,y} \sum_{m=1}^{\infty} \mathbb{E}_y\left(\varphi(X_{m-1})\mathbb{1}_{\{\tau_D=m-1\}}\right)$$

$$= \sum_{y\in S} P_{x,y}\, \mathbb{E}_y\left(\varphi(X_{\tau_D})\mathbb{1}_{\{\tau_D<\infty\}}\right) = \sum_{y\in S} P_{x,y} f(y) = Pf(x)$$

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What about the uniqueness of solution of DP?

Counterexample: Consider DP: $\begin{cases} Pf(x) = f(x), & x\in D \\ 1, & x \in D^c \end{cases}$

One can show (as in the above proof), that $f(x) := \mathbb{P}_x(\tau_D < \infty)$ is a solution. But if $f(x) < 1$ for some x (excludes irreducible, recurrent MC) then the solution is not unique ! as f and 1 are both solutions.

Theorem [cond. for uniq.]: If $\mathbb{P}_x(\tau_D = +\infty) = 0 \quad \forall x \in D$ then (DP) has a unique bounded solution of (DP), given by ⊛.

A main tool for the proof of the uniqueness of solution of the (DP) - and not only - is the following lemma.

Lemma: (Lévy's Martingale)

Let X be a MC with transition matrix P and $g: S\to\mathbb{R}$ bounded.

$$\Longrightarrow \quad M_n^g := g(X_n) - g(X_0) - \sum_{k=0}^{n-1}\left(\underbrace{Pg(X_k) - g(X_k)}_{Lg(X_k)}\right), \quad n\in\mathbb{N}$$

is a martingale w.r.t. $\mathcal{F}_n^X = \sigma(X_k,\, k\le n)$.

Proof (of lemma):

- $M_n^g \in L^1$ by boundedness of g.
- $$\mathbb{E}\left(M_{n+1}^g \mid \mathcal{F}_n^X\right) = \mathbb{E}\left[g(X_{n+1}) \mid \mathcal{F}_n^X\right) - g(X_0) - \underbrace{\sum_{k=0}^{n}\left(Pg(X_k) - g(X_k)\right)}_{\in \mathcal{F}_n^X} \overset{*}{=}$$

Since $\mathbb{E}\left[g(X_{n+1}) \mid \mathcal{F}_n^X\right) = \mathbb{E}_{X_n}\left(g(X_1)\right) = Pg(X_n)$, which cancels with one term in the sum, we get

$$\overset{*}{=} g(X_n) - g(X_0) - \sum_{k=0}^{n-1}\left(Pg(X_k) - g(X_k)\right) = M_n^g \qquad \#$$

Proof of Theorem [cond. for uniq]

Consider the martingale $(M_n^f)_{n \in \mathbb{N}}$, where f is a (general) bounded sol. of (DP), and the corresponding stopped martingale at time $\tau_D = \tau$ (for convenience):

$$M_n^\tau = f(X_{\tau \wedge n}) - f(X_0) - \underbrace{\sum_{k=0}^{(n-1)\wedge(\tau-1)}\left(Pf(X_k) - f(X_k)\right)}_{=0}$$

- The sum $= 0$, since $\forall k \leq \tau - 1$, $X_k \in D$ and hence $Pf(X_k) - f(X_k) = 0$ by harmonicity of f on D.
- Taking the average w.r.t. $\mathbb{P}_x$, with $x \in D$, we get

$$0 = \mathbb{E}_x\left(M_n^\tau\right) = \mathbb{E}_x\left(f(X_{\tau \wedge n})\right) - f(x)$$

$$\Longrightarrow f(x) = \mathbb{E}_x\left(f(X_{\tau \wedge n})\right)$$

Taking the limit $n \to \infty$, $X_{\tau \wedge n} \xrightarrow[n\to\infty]{} X_\tau \in D^c$ a.s. (τ is a.s. finite) and using that $f(x) = \varphi(x)$ $\forall x \in D^c$, and we obtain

$$f(x) = \mathbb{E}_x\left(\varphi(X_\tau)\right),$$

providing the uniqueness of the solution.

$g(x) = \mathbb{E}_x(\varphi(X_\tau))$,

providing the uniqueness of the solution #

A similar representation can be given for the solutions of the Poisson equation, for $h: D \to \mathbb{R}$ bounded

$$(PE) \quad \begin{cases} Pf(x) - f(x) = -h(x) & \text{if } x \in D \iff Lf(x) = -h(x) \\ f(x) = 0 & \text{if } x \in D^c \end{cases}$$

If $\mathbb{P}_x(\tau_D < \infty) = 1$, $\forall x \in D$, then $\exists$ at most* a bounded solution of (PE) given by

$$\boxed{f(x) = \mathbb{E}_x\left(\sum_{n=0}^{\tau_D - 1} h(X_n)\right)}$$

* to ensure the existence of a bounded solution we need extra-assumption, e.g. $\mathbb{E}_x(\tau_D) < \infty$ $\forall x \in D$

Verify as an exercise trough the following steps:

* For $A \subset S$, set $G_A(x,y) = \mathbb{E}_x(N_{\tau_A}(y))$, $\forall x, y \in S$

Then notice that the sol. above can be written as

$$f(x) = G_D h(x) = \sum_{y \in S} G_D(x,y) h(y) \quad \text{(check!)}$$

* In particular, under the hypotheses that $\mathbb{E}_x(\tau_D) < \infty$ $\forall x \in D$ (so that G_D is bounded), we get

$$1.\ LG_D(x,y) = \sum_{z \in S} \underbrace{L(x,z)}_{P_{x,z} - \delta_x(z)} G_D(z,y) = -\delta_x(y) \quad \text{(check!)}$$

$$2.\ Lf(x) = LG_D h(x) = \sum_{y \in S} LG_D(x,y) h(y) = -h(x). \ \#$$

The function $G_D(x,y)$ is then precisely the Green function on D

The function $G_D(x,y)$ is then precisely the Green function on D associated the the differential operator $L = P - Id$ (generator).

Before enter a specific model of Markov chain, let us discuss the case of MC on S with absorbing states.

Lemma: Let X be a MC on S finite, and let $A \subset S$ be the set of absorbing states: $A = \{x \in S : P_{x,x} = 1\}$

Assume that $\forall x \in S$, $\mathbb{P}_x(\exists n \geq 0 : X_n \in A) > 0$

Then, $\forall x \in S$: $\boxed{\mathbb{P}_x(\exists n \geq 0 : X_n \in A) = 1}$ (X is absorbed in A a.s.)

(Recall the Wright model and the voter model → lectures 2-3)

Proof:

Let $T_A = \inf\{n \geq 0 : X_n \in A\}$, so that

$$\{\exists n \geq 0 : X_n \in A\} = \{T_A < \infty\}.$$

By hypothesis, $\forall x \in S$, $\exists\, \varepsilon_x > 0$ and $N_x \in \mathbb{N}$ s.t.

$$\mathbb{P}_x(T_A \leq N_x) > \varepsilon_x .$$

Since S is finite, we can take $N = \max_{x \in S} N_x$, $\varepsilon = \min_{x \in S} \varepsilon_x$ and conclude that

$$\forall x \in S : \mathbb{P}_x(T_A \leq N) > \varepsilon \iff \mathbb{P}_x(T_A > N) \leq 1 - \varepsilon . \quad (*)$$

Using iteratively the strong Markov property and the bound $(*)$, we get, $\forall k \in \mathbb{N}$:

$$\mathbb{P}_x(T_A > kN) \leq (1-\varepsilon) \max_{y \in S} \mathbb{P}_y(T_A > (k-1)N) \leq \dots$$

$$\mathbb{P}_x(T_A > kN) \le (1-\varepsilon)\max_{y\in S} \mathbb{P}_y(T_A > (k-1)N) \le \dots$$

$$\le (1-\varepsilon)^k$$

But $\mathbb{P}_x(T_A = \infty) = \lim_{k\to\infty} \mathbb{P}(T_A > kN) \le \lim_{k\to\infty} (1-\varepsilon)^k = 0$ #

The case $|S| = +\infty$ includes instead different situations, and we may have $\mathbb{P}_x(T_A = \infty) = 0$ or > 0.
The branching process is indeed an example of MC on $\mathbb{N}$ with an absorbing state (0) where the MC can be absorbed with 1 or < 1 probability depending on one parameter (next lecture!).