LEZIONE 12

CURVE (parametriche)

piane: $\alpha : J \longrightarrow \mathbb{R}^2$ t.c. $\underbrace{\|\alpha'(t)\| = 1}_{\text{velocità unitaria}}$

$t \longmapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

di classe $\mathcal{C}^\infty$

Base di Frenet: $\begin{cases} \underline{t} = \alpha' = \begin{pmatrix} x' \\ y' \end{pmatrix} & \text{versore tangente} \\ \underline{n} = \text{vettore ortonormale ad } \alpha' \text{ t.c. base sia equiorientata} = \begin{pmatrix} -y' \\ x' \end{pmatrix} & \text{versore normale} \end{cases}$

Equazioni di Frenet: $\begin{cases} \underline{t}' = k \cdot \underline{n} \\ \underline{n}' = -k\, \underline{t} \end{cases} \rightsquigarrow \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} k x' \\ k y' \end{pmatrix}$

k : curvatura

$\rightsquigarrow \underline{t}' \cdot \underline{n} = k \cdot \underline{t} \cdot \underline{n} = k$

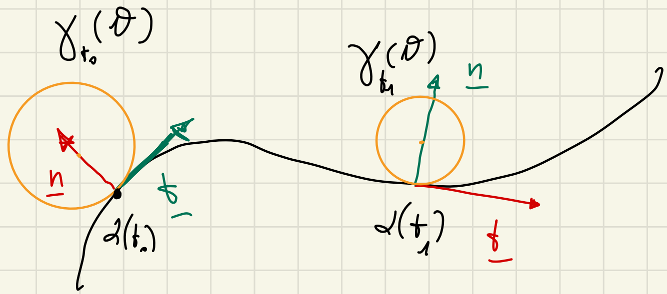
(•) $k(t_0) = 0 \rightsquigarrow t_0$ detto flesso

(•) $k(t_0) \neq 0 \rightsquigarrow \left| \dfrac{1}{k(t_0)} \right|$ raggio di curvatura (in t_0)

$c(t_0) = \alpha(t_0) + \dfrac{1}{k(t_0)} \underline{n}(t_0)$ centro di curvatura (in t_0)

$$\gamma_t(\theta) = c(t_0) + \frac{1}{k(t_0)} (\cos\theta)\cdot \underline{t} + \frac{1}{k(t_0)} (\text{sen}\,\theta)\,\underline{n}$$

cerchio osculatore in t_0



$$(x'^2 + y'^2)' = 2x'\cdot x'' + 2y'y''$$

nello spazio:

$\alpha : J \to \mathbb{R}^3 \qquad \|\alpha'(t)\| = 1$

nota che $\underline{n}^{(2)} \neq \underline{n}^{(3)}$

$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ di classe C^∞

Base di Frenet:

$$\left\{ \begin{array}{ll} \underline{t} = \alpha' & \text{versore tangente} \\ \underline{n} = \dfrac{\alpha''}{\|\alpha''\|} & \text{versore normale} \\ \underline{b} = \underline{t} \times \underline{n} & \text{versore bitangente} \end{array} \right.$$

Oss Questo perché se $1 = \|\alpha'\|$

$\Rightarrow (\|\alpha'\|^2)' = 2\alpha'\cdot\alpha'' = 0 \Rightarrow \alpha' \perp \alpha''$

Equazioni di Frenet :

$$\begin{cases} \underline{t}' = k\,\underline{n} \\ \underline{n}' = -k\,\underline{t} + \tau\,\underline{b} \\ \underline{b}' = -\tau\,\underline{n} \end{cases}$$

$\rightsquigarrow k^{(3)} = \|\alpha''\| \geq 0$

in modulo $k^{(2)} \equiv k^{(3)}$

ma possono avere segno opposto

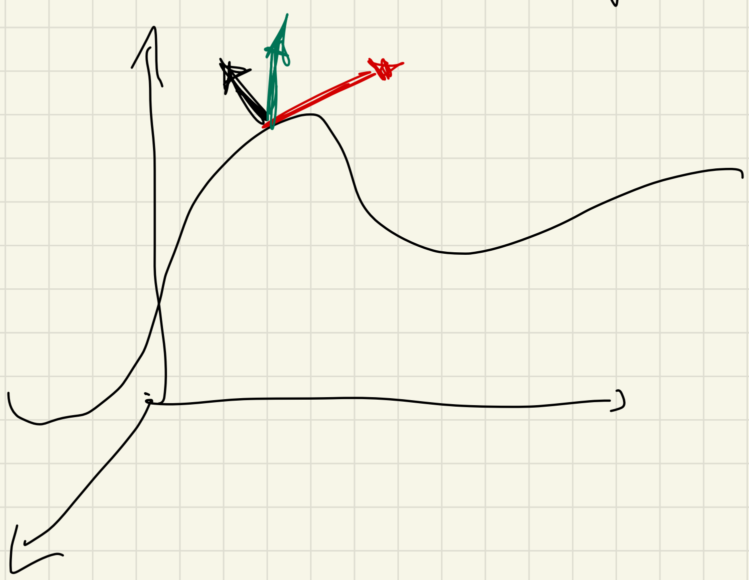
1° piano osculatore $\langle \underline{t}, \underline{n} \rangle$

piano rettificante $\langle \underline{t}, \underline{b} \rangle$

piano normale osculatore $\langle \underline{n}, \underline{b} \rangle$

piano osculatore affine $\alpha(t) + \langle \underline{t}, \underline{n} \rangle$

cerchio osculatore affine stesse formule come sopra : contenuto nel piano osculatore affine



CURVE in $\mathbb{R}^3$ e velocità qualsiasi

$$\alpha : I \longrightarrow \mathbb{R}^3 \qquad \mathcal{C}^\infty$$
$$t \longmapsto \alpha(t)$$

α', α'' sono linearm. indip.

$$\beta : J \longrightarrow \mathbb{R}^3$$
$$s \longmapsto \beta(s)$$

riparam. a velocità unitaria

$\theta : t \mapsto s$

sia $\alpha = \beta \circ \theta$, con θ diffeom di classe $\mathcal{C}^\infty$

Prop Se α', α'' linearmente indipendenti posso trovare una base di Frénet:

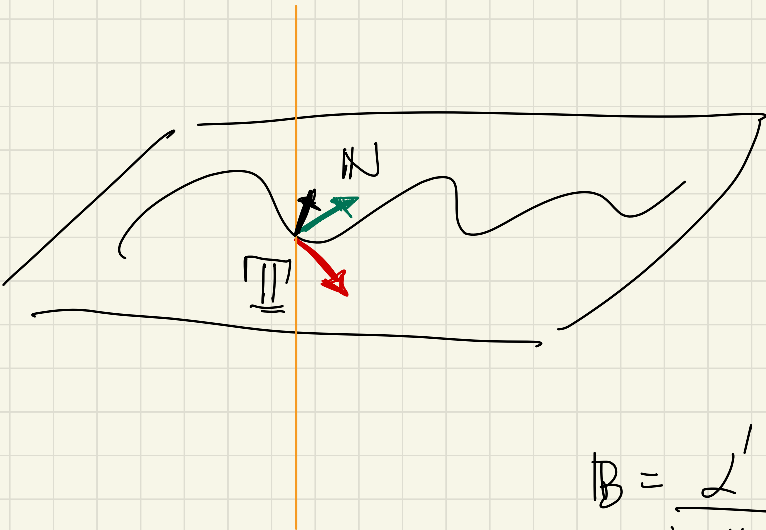
$$\mathbb{T} = \frac{\alpha'}{\|\alpha'\|} \qquad K = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$$

$$\mathbb{N} = \mathbb{B} \times \mathbb{T}$$

$$\mathbb{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} \qquad T = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$$

OSS Nel caso di curve in $\mathbb{R}^2$ si possono trovare delle formule immergendole in $\mathbb{R}^3$:
si può allora trovare base di Frenet sse $\alpha'(t) \neq 0$

α non è velocità unitaria



$N = \alpha''_{\perp}$

$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$$

$\{\alpha', \alpha''\}$ formano una base equiorientata alla base canonica

$$\Rightarrow B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow k^{(2)} > 0$$

se $\{\alpha', \alpha''\}$ non formano una base equiorientata alla canonica

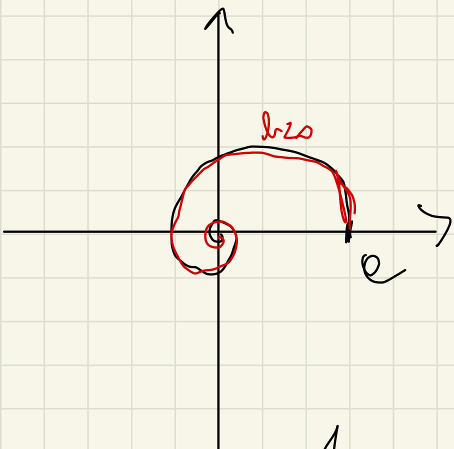
$$\Rightarrow B = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \Leftrightarrow k^{(2)} < 0$$

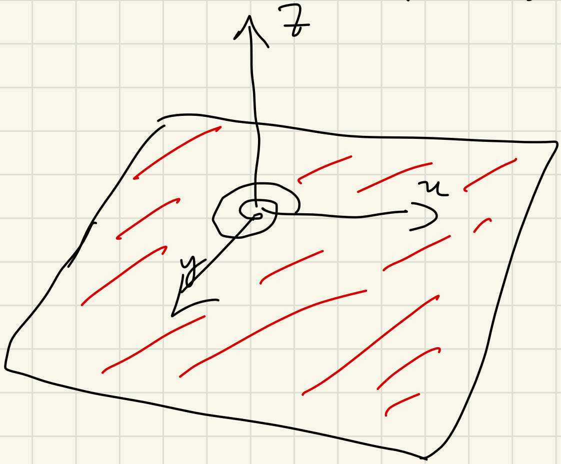
AD ESEMPIO L'ALTRA VOLTA AVEVAMO:

$$\boxed{a>0,\quad b<0}$$

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$t \longmapsto a\, e^{bt+it} = a\, e^{bt} \begin{pmatrix} \cos t \\ \operatorname{sen} t \end{pmatrix}$$





$$k_i = \frac{1}{a\, e^{bt} \sqrt{b^2+1}} \quad \text{come curva in } \mathbb{R}^3$$

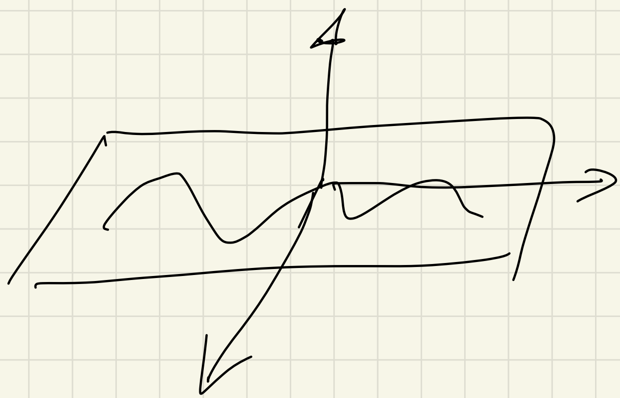
$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} = \frac{\cancel{a^2 e^{2bt}}}{\cancel{a^2 e^{2bt}} \cdot (b^2+1)} \begin{pmatrix} 0 \\ 0 \\ b^2+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ curvatura positiva come curva in $\mathbb{R}^2$!

E SE LA CURVATURA CAMBIASSE DI SEGNO ? $\mathbb{R}^3$ $\mathbb{R}^2$

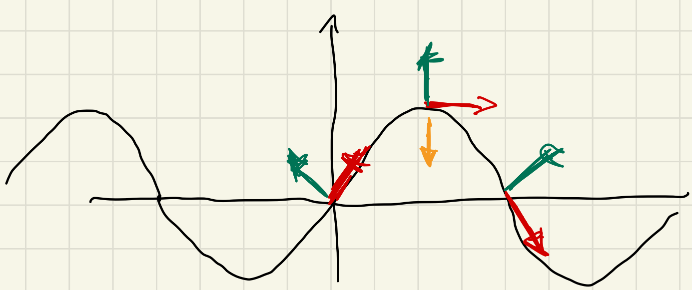
$$\gamma = \begin{pmatrix} t \\ \operatorname{sen} t \end{pmatrix}$$

$$\gamma' = \begin{pmatrix} 1 \\ \cos t \end{pmatrix}$$



$$\boxed{T}^{(2)} = \frac{1}{\sqrt{1+\cos^2 t}} \begin{pmatrix} 1 \\ \cos t \end{pmatrix}$$

$$N^{(2)} = \frac{1}{\sqrt{1+\cos^2 t}} \begin{pmatrix} -\cos t \\ 1 \end{pmatrix}$$



$t=0$ $\quad \boxed{T} \overset{\times}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$N^{(2)} \overset{\times}{=} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$t = \frac{\pi}{2}$ $\quad \boxed{T} \overset{\times}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$N^{(2)} \overset{\times}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$t = \pi$ $\quad \boxed{T} \overset{\times}{=} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$N^{(2)} \overset{\times}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\gamma'' = \begin{pmatrix} 0 \\ -\operatorname{sen} t \end{pmatrix} \qquad t=0 \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$t = \frac{\pi}{2} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\gamma' \times \gamma'' = \det \begin{pmatrix} e_1 & 1 & 0 \\ e_2 & \cos t & -\operatorname{sen} t \\ e_3 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ -\operatorname{sen} t \end{pmatrix}$$

$$\mathbb{B} = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{se} \quad \pi + 2k\pi < t < 2(k+1)\pi \\ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} & \text{se} \quad 2k\pi < t < (2k+1)\pi \end{cases}$$

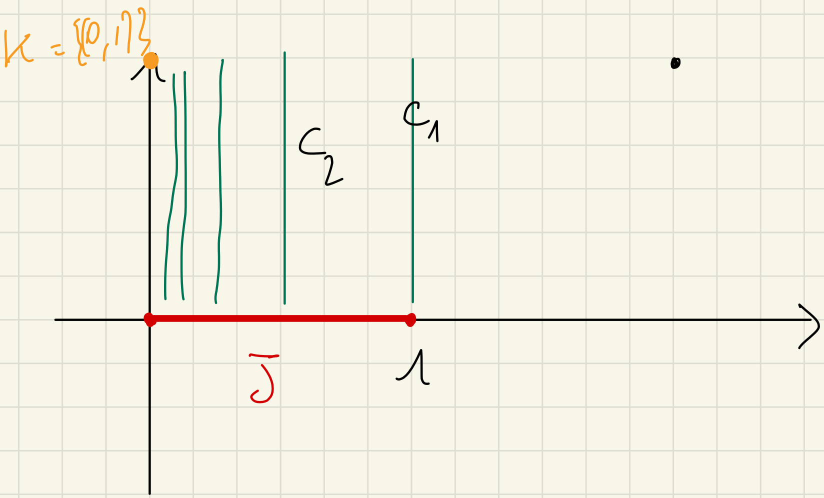
$$K^{(2)} \begin{cases} > 0 & \text{se} \quad \pi + 2k\pi < t < 2(k+1)\pi \\ < 0 & \text{se} \quad 2k\pi < t < (2k+1)\pi \end{cases}$$

$$K^{(3)} = \frac{\| \gamma' \times \gamma'' \|}{\| \gamma' \|^3} = \frac{|\mathrm{sen}\, t|}{\left(\sqrt{1 + \cos^2 t}\right)^3}$$

$$K^{(2)} = \begin{cases} \dfrac{|\mathrm{sen}\, t|}{\left(\sqrt{1 + \cos^2 t}\right)^3} & \text{se } \pi + 2k\pi < t < 2(k+1)\pi \\[2ex] -\dfrac{|\mathrm{sen}\, t|}{\left(\sqrt{1 + \cos^2 t}\right)^3} & \text{se } 2k\pi < t < (2k+1)\pi \end{cases}$$

Esercizio 2. Per ogni $n \geq 1$ intero, definire $C_n = \{(1/n, y) | y \in [0,1]\}$, $J = \{(x,0) | x \in [0,1]\}$, $K = \{(0,1)\}$, $X = J \cup K \cup (\cup_{n\geq 1} C_n)$. Mostrare che X è connesso ma non è connesso per archi.

Esercizio 1. Sia $A = \{(0,y) | y \in [-1,1]\}$, $B = \{(x, \sin(1/x)) | s \in (0,1]\}$, $X = A \cup B$. Mostrare che X è connesso ma non è connesso per archi.



$(0,1)$ é punto di accumulazione per l'insieme $J \cup \left(\bigcup_{n \geq 1} C_n \right) = Y$

la chiusura di Y in X é tutto X

$$\overline{Y}^X = X$$

RICORDARE

$$Y \subseteq S \subseteq \overline{Y}^{X} \Rightarrow S \text{ connesso}$$

Y connesso $\Uparrow$ Connesso per archi

in particolare

$$X = \overline{Y}^{X} \text{ connesso}$$

Alternativa

Osservo sempre che Y connesso per archi $\Rightarrow$ Connesso

Supponiamo $X \subseteq A \overset{\circ}{\cup} B$ aperti di $\mathbb{R}^2$

Supponiamo $(0,1) \in A \Rightarrow A \cap Y \neq \emptyset$

(scelgo $n \gg 0 \Rightarrow (\frac{1}{n}, y) \in A$ per $y \sim 1$)

$$Y = X \cap Y = \underbrace{(A \cap Y)}_{\neq \emptyset} \overset{\circ}{\cup} (B \cap Y) \Rightarrow B \cap Y = \emptyset$$

perché Y connesso

$$\left.\begin{array}{l} B \cap Y = \emptyset \\ B \not\ni (0,1) \end{array}\right\} B \cap X = \emptyset$$

$\Rightarrow X$ connesso

$$\gamma : [0,1] \longrightarrow X \subseteq \mathbb{R}^2$$
$$t \longmapsto \gamma(t) \in X$$

continuo

t.c. $\gamma(0) = (0,1)$, $\gamma(1) \in Y$

$\overset{?}{\Rightarrow} \lightning$

$$t_0 = \sup\{t : \gamma(t) = (0,1)\} \neq 1$$
$$= \max\{t : \gamma(t) = (0,1)\}$$

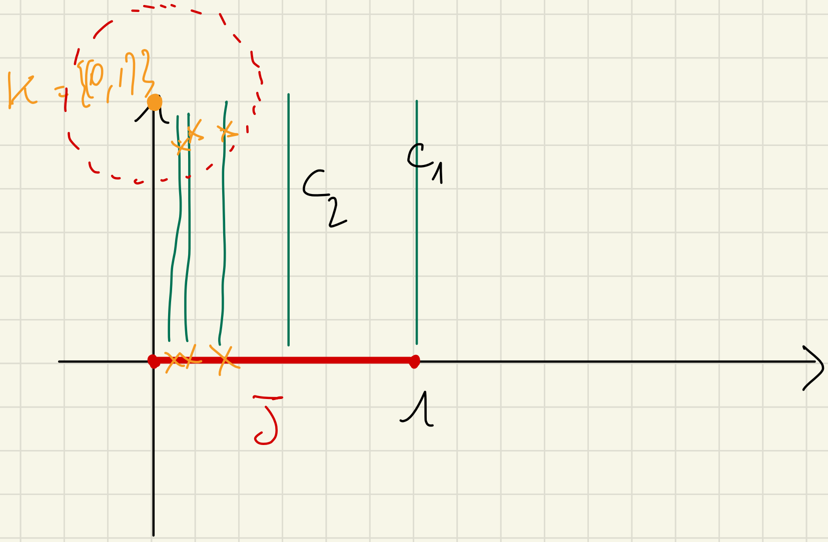
$\{t_n\}_{n>1} \nearrow t_0 \underset{\gamma \text{ contin}}{\Rightarrow} \gamma(t_n) \longrightarrow \gamma(t_0)$

$t_n \in [0,1]$, $\gamma(t_n) = (0,1)$, $\gamma(t_0) = (0,1)$

γ continua in $t_0 \Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0$

t.c. $\forall t \in (t_0 - \delta, t_0 + \delta) \Rightarrow \gamma(t) \in D((0,1), \varepsilon)$

$\in (t_0, t_0 + \delta)$



P_1 continua $\Leftarrow$) $p_1 \circ \gamma$ continua

$\Rightarrow P_1(\gamma(t_0, t_0+\delta))$ connesso

$\Rightarrow \ = \begin{cases} \text{pto} & ↯ \\ \text{intervallo} : \text{NO} & ↯ \end{cases}$ ↯