LEZIONE 11

CURVE (parametriche)

piane :
$$\alpha : J \longrightarrow \mathbb{R}^2 \qquad \text{t.c.} \quad \underbrace{\|\alpha'(t)\| = 1}_{\text{velocità unitaria}}$$
$$t \longmapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

di classe $\mathcal{C}^\infty$

Base di Frenet :
$$\begin{cases} \underline{t} = \alpha' = \begin{pmatrix} x' \\ y' \end{pmatrix} & \text{versore tangente} \\ \underline{n} = \text{vettore ortonormale ad } \alpha' \text{ t.c. base sia equiorientata} = \begin{pmatrix} -y' \\ x' \end{pmatrix} & \text{versore normale} \end{cases}$$

Equazioni di Frenet :
$$\begin{cases} \underline{t}' = k \cdot \underline{n} \\ \underline{n}' = -k\, \underline{t} \end{cases} \rightsquigarrow \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} k x' \\ k y' \end{pmatrix}$$

k : curvatura
$$\rightsquigarrow \underline{t}' \cdot \underline{n} = k \cdot \underline{t} \cdot \underline{n} = k$$

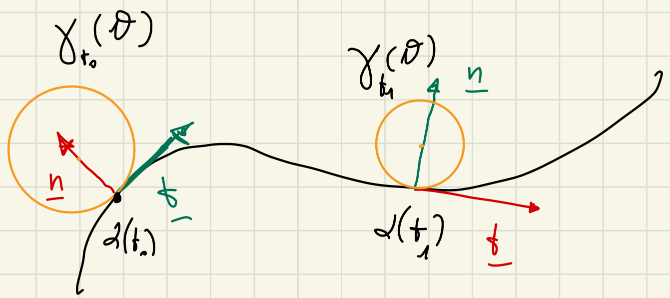
(•) $k(t_0) = 0 \rightsquigarrow t_0$ detto flesso

(•) $k(t_0) \neq 0 \rightsquigarrow \left| \dfrac{1}{k(t_0)} \right|$ raggio di curvatura (in t_0)

$$c(t_0) = \alpha(t_0) + \frac{1}{k(t_0)}\, \underline{n}(t_0)$$ centro di curvatura (in t_0)

$$\gamma_{t}(\theta) = c(t_0) + \frac{1}{k(t_0)} (\cos\theta) \cdot \underline{t} + \frac{1}{k(t_0)} (\operatorname{sen}\theta)\, \underline{n}$$

cerchio osculatore in t_0



$$(x'^2 + y'^2)' = 2x' \cdot x'' + 2y'y''$$

nello spazio:

$$\alpha : J \to \mathbb{R}^3 \qquad \|\alpha'(t)\| = 1$$

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \text{di classe } \mathcal{C}^\infty$$

Base di Frenet:

$$\begin{cases} \underline{t} = \alpha' & \text{versore tangente} \\ \underline{n} = \dfrac{\alpha''}{\|\alpha''\|} & \text{versore normale} \\ \underline{b} = \underline{t} \times \underline{n} & \text{versore binormale} \end{cases}$$

Oss Questo perché se $1 = \|\alpha'\|$

$$\Rightarrow (\|\alpha'\|^2)' = 2\alpha' \cdot \alpha'' = 0 \Rightarrow \alpha' \perp \alpha''$$

Equazioni di Frenet :

$$\begin{cases} \underline{t}' = k\,\underline{n} \\ \underline{n}' = -k\,t + \tau\,\underline{b} \\ \underline{b}' = -\tau\,\underline{n} \end{cases}$$

k: curvatura

τ: torsione

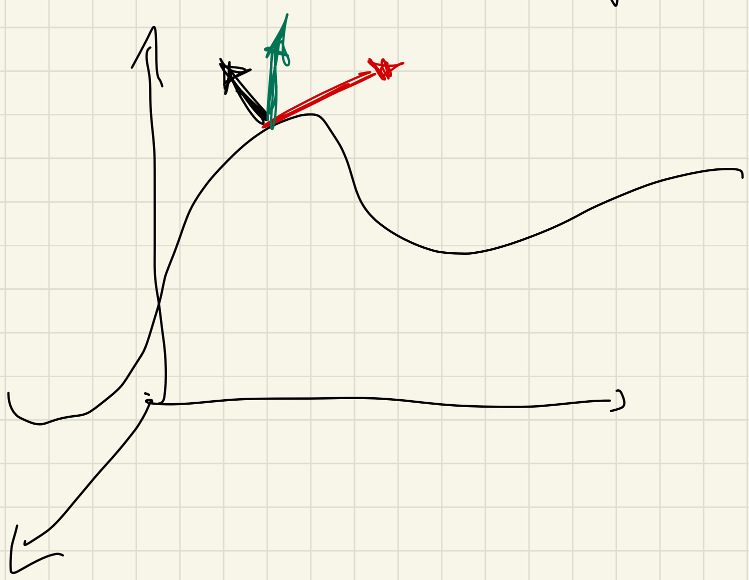
1° piano osculatore $\langle \underline{t}, \underline{n} \rangle$

piano rettificante $\langle \underline{t}, \underline{b} \rangle$

piano normale osculatore $\langle \underline{n}, \underline{b} \rangle$

piano osculatore affine $\alpha(t) + \langle \underline{t}, \underline{n} \rangle$

cerchio osculatore affine stesse formule come sopra : contenuto nel piano osculatore affine



CURVE in $\mathbb{R}^3$ e velocità qualsiasi

$$\alpha : I \longrightarrow \mathbb{R}^3 \qquad \mathcal{C}^\infty$$
$$t \longmapsto \alpha(t)$$

α', α'' sono linearm. indip.

$$\beta : J \longrightarrow \mathbb{R}^3$$
$$s \longmapsto \beta(s)$$

riparam. a velocità unitaria

$\vartheta : t \mapsto s$

sia $\alpha = \beta \circ \vartheta$, con ϑ diffeom di classe $\mathcal{C}^\infty$

Prop Se α', α'' linearmente indipendenti posso trovare una base di Frénet:

$$\mathbb{T} = \frac{\alpha'}{\|\alpha'\|} \qquad K = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$$

$$\mathbb{N} = \mathbb{B} \times \mathbb{T}$$

$$\mathbb{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} \qquad T = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$$

OSS Nel caso di curve in $\mathbb{R}^2$ si possono trovare delle formule immergendole in $\mathbb{R}^3$: si può allora trovare base di Frenet sse $\alpha'(t) \neq 0$

$$\mathbb{T} = \frac{\alpha'}{\|\alpha'\|} \qquad K = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$$

$$\mathbb{N} = \mathbb{B} \times \mathbb{T}$$

$$\mathbb{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} \qquad T = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$$

α curva in $\mathbb{R}^3$ t.c. α', α'' siano linearm. indipendenti

$\boxed{\text{ES 4}}$

$$\sigma(t) = \begin{pmatrix} 3t - t^3 \\ 3t^2 \\ 3t + t^3 \end{pmatrix}$$

$$\sigma' = \begin{pmatrix} 3 - 3t^2 \\ 6t \\ 3 + 3t^2 \end{pmatrix} = 3 \begin{pmatrix} 1 - t^2 \\ 2t \\ 1 + t^2 \end{pmatrix}$$

$$\|\sigma'\| = 3 \sqrt{(1-t^2)^2 + 4t^2 + (1+t^2)^2} =$$

$$= 3 \cdot \sqrt{2 + 2t^4 + 4t^2} =$$

$$= 3 \cdot \sqrt{2} \cdot \sqrt{t^4 + 2t^2 + 1} =$$

$$= 3 \cdot \sqrt{2} \cdot (t^2 + 1)$$

$$\boxed{\underline{!!}} = \cancel{3} \begin{pmatrix} 1 - t^2 \\ 2t \\ 1 + t^2 \end{pmatrix} \cdot \frac{1}{\cancel{3} \cdot \sqrt{2} \cdot (t^2+1)} =$$

$$= \frac{1}{t^2+1} \begin{pmatrix} 1 - t^2 \\ 2t \\ 1 + t^2 \end{pmatrix}$$

$$v''' = 6 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$v'' = 3 \begin{pmatrix} -2t \\ 2 \\ 2t \end{pmatrix} = 6 \begin{pmatrix} -t \\ 1 \\ t \end{pmatrix}$$

$$v' \times v'' = 6 \cdot 3 \det \begin{pmatrix} e_1 & 1 - t^2 & -t \\ e_2 & 2t & 1 \\ e_3 & 1 + t^2 & t \end{pmatrix} =$$

$$= 18 \begin{pmatrix} 2t^2 - 1 - t^2 \\ -[1 - t^2 + 2t^2] \\ 1 - t^2 + 2t^2 \end{pmatrix} =$$

$$= 18 \begin{pmatrix} t^2 - 1 \\ -(1 + t^2) \\ t^2 + 1 \end{pmatrix}$$

$$\sqrt{(t^2-1)^2 + (t^2+1)^2 + (t^2+1)^2} =$$

$$= \sqrt{t^4 + 1 - \cancel{2t^2} + t^4 + 1 + \cancel{2t^2} + t^4 + 1 + 2t^2} =$$

$$= \sqrt{3t^4 + 3 + 2t^2}$$

$$\mathbb{B} = \frac{1}{\sqrt{3t^4 + 2t^2 + 3}} \begin{pmatrix} t^2 - 1 \\ -(t^2+1) \\ t^2 + 1 \end{pmatrix}$$

$$\mathbb{N} = \mathbb{B} \times \underline{\mathbb{T}} = \frac{1}{(t^2+1) \cdot \sqrt{3t^4 + 2t^2 + 3}} \cdot$$

$$\cdot \det \begin{pmatrix} e_1 & t^2 - 1 & 1 - t^2 \\ e_2 & -(t^2+1) & 2t \\ e_3 & t^2 + 1 & 1 + t^2 \end{pmatrix} =$$

$$= \frac{1}{(t^2+1)\cdot\sqrt{3t^4+2t^2+3}} \cdot \begin{pmatrix} -t^4-1-2t^2-2t^3-2t \\ -\left[t^4-1-(1-t^4)\right] \\ 2t^3-2t+1-t^4 \end{pmatrix}$$

$$= \frac{1}{(t^2+1)\cdot\sqrt{3t^4+2t^2+3}} \cdot \begin{pmatrix} -t^4-2t^3-2t^2-2t-1 \\ 2-2t^4 \\ -t^4+2t^3-2t+1 \end{pmatrix}$$

$$K = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{18\cdot\sqrt{3t^4+3+2t^2}}{\left(3\cdot\sqrt{2}\cdot(t^2+1)\right)^3} =$$

$$= \frac{\overset{\overset{1}{\cancel{9}}}{\cancel{18}}\sqrt{3t^4+3+2t^2}}{\underset{3}{\cancel{27}}\cdot\cancel{2}\sqrt{2}\cdot(t^2+1)^3} = \frac{\sqrt{3t^4+2t^2+3}}{3\sqrt{2}\,(t^2+1)^3}$$

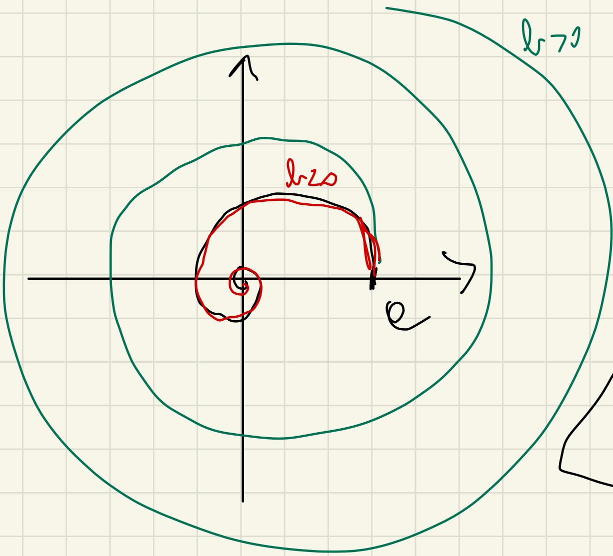
$$T = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2} = \frac{18 \cdot 6 \begin{pmatrix} t^2 - 1 \\ -(1 + t^2) \\ t^2 + 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}{\left(18 \cdot \sqrt{3t^4 + 3 + 2t^2}\right)^2} =$$

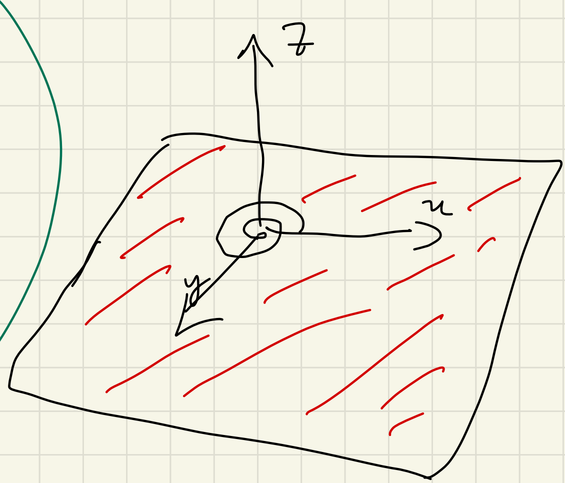
$$= \frac{\cancel{18} \cdot \cancel{6}^{1} \cdot (1 - \cancel{t^2} + \cancel{t^2} + 1)}{\cancel{18} \cdot \underset{3}{\cancel{18}} \, (3t^4 + 2t^2 + 3)} =$$

$$= \frac{2}{3} \cdot \frac{1}{3t^4 + 2t^2 + 3}$$

$\boxed{\text{ES 3}}$ Fissiamo $c > 0$, $b < 0$

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto c\, e^{bt+it} = c\, e^{bt} \begin{pmatrix} \cos t \\ \operatorname{sen} t \end{pmatrix}$$





$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \longmapsto c\, e^{bt} \begin{pmatrix} \cos t \\ \operatorname{sen} t \\ 0 \end{pmatrix}$$

$$\underline{T} = \frac{\gamma'}{\|\gamma'\|} \rightsquigarrow \mathbb{N}$$

K

$$v' = \begin{pmatrix} a e^{bt} \cdot b \cos t - a e^{bt} \cdot \operatorname{sen} t \\ a e^{bt} \cdot b \cdot \operatorname{sen} t + a e^{bt} \cdot \cos t \end{pmatrix}$$

$$= a e^{bt} \begin{pmatrix} b \cos t - \operatorname{sen} t \\ b \operatorname{sen} t + \cos t \end{pmatrix}$$

$$v'' = \begin{pmatrix} a e^{bt} b \cdot (b \cos t - \operatorname{sen} t) + a e^{bt} (-b \operatorname{sen} t + \cos t) \\ a e^{bt} b \cdot (b \operatorname{sen} t + \cos t) + a e^{bt} (b \cos t - \operatorname{sen} t) \end{pmatrix}$$

$$= a e^{bt} \begin{pmatrix} b^2 \cos t - 2 b \operatorname{sen} t - \cos t \\ b^2 \operatorname{sen} t + 2b \cos t - \operatorname{sen} t \end{pmatrix}$$

$$\|v'\| = a e^{bt} \sqrt{b^2 \cos^2 t + \operatorname{sen}^2 t - \cancel{2b \cos t \operatorname{sen} t} + b^2 \operatorname{sen}^2 t + \cos^2 t + \cancel{2b \operatorname{sen} t \cos t}}$$

$$= a e^{bt} \sqrt{b^2 + 1}$$

$$\underline{\mathbb{T}} = \frac{\cancel{e^{bt}}}{\cancel{e^{bt}} \cdot \sqrt{b^2+1}} \begin{pmatrix} b\cos t - \operatorname{sen} t \\ b \operatorname{sen} t + \cos t \end{pmatrix} =$$

$$= \frac{1}{\sqrt{b^2+1}} \begin{pmatrix} b\cos t - \operatorname{sen} t \\ b \operatorname{sen} t + \cos t \end{pmatrix}$$

$$\mathbb{N} = \frac{1}{\sqrt{b^2+1}} \begin{pmatrix} -b\operatorname{sen} t - \cos t \\ b\cos t - \operatorname{sen} t \end{pmatrix}$$

$$\det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = 1$$

$$x^2 + y^2 = 1$$

$$K = \frac{\| \alpha' \times \alpha'' \|}{\| \alpha' \|^3} =$$

$$v' \times v'' = e\, e^{bt} \cdot \theta\, e^{bt}$$

$$= \det \begin{pmatrix} e_1 & b\cos t - \operatorname{sen} t & b^2 \cos t - 2b \operatorname{sen} t - \cos t \\ e_2 & b \operatorname{sen} t + \cos t & b^2 \operatorname{sen} t + 2b\cos t - \operatorname{sen} t \\ e_3 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ \cancel{b^3 \operatorname{sen} t \cos t} + 2b^2 \cos^2 t - \cancel{b \operatorname{sen} t \cos t} + \\ - b^2 \operatorname{sen}^2 t - \cancel{2b \overset{\operatorname{sen} t}{\cos t}} + \operatorname{sen}^2 t + \\ - \left(\cancel{b^3 \operatorname{sen} t \cos t} - 2b^2 \operatorname{sen}^2 t - \cancel{b \operatorname{sen} t \cos t} \right. \\ \left. + b^2 \cos^2 t - \cancel{2b \operatorname{sen} t \cos t} - \cos^2 t \right) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2b^2 - b^2 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b^2 + 1 \end{pmatrix}$$

$$\| \sigma' \times \sigma'' \| = a^2 \cdot e^{2bt} \cdot b^2 + 1$$

$$K = \frac{\cancel{a^2} \cdot \cancel{e^{2bt}} \cdot \cancel{(b^2+1)}}{a^{\cancel{3}} e^{\cancel{3}bt} \sqrt{b^2+1} \cdot \cancel{(b^2+1)}} =$$

$$= \frac{1}{a e^{bt} \sqrt{b^2+1}}$$

QUADRICHE COME VARIETÀ DIFFERENZIALI (IMMERSE)

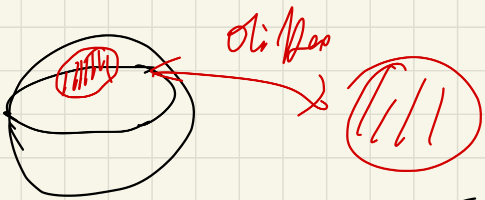
- $S^2: \; x^2+y^2+z^2=R^2$

$$\mathbb{R}^2 \xrightarrow{\sigma} \mathbb{R}^3$$

$$(\theta,\varphi) \longmapsto R\begin{pmatrix} \operatorname{sen}\theta \, \operatorname{sen}\varphi \\ \cos\theta \, \operatorname{sen}\varphi \\ \cos\varphi \end{pmatrix}$$

$\operatorname{im}\sigma = \{x \in \mathbb{R}: \|x\| = R\}$
sfera

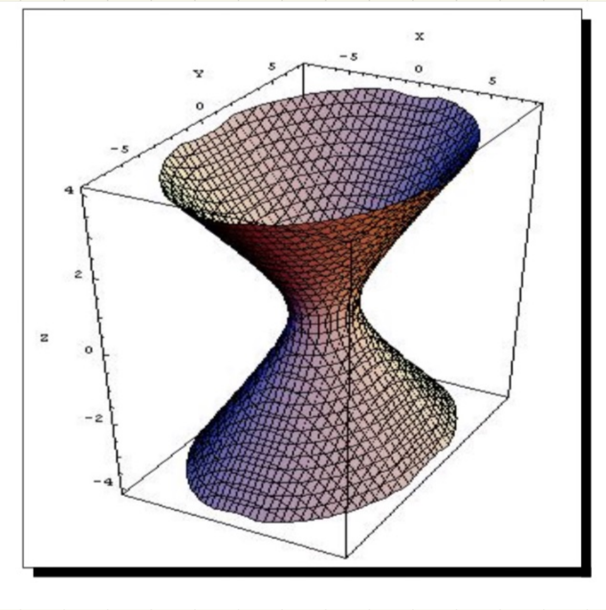
l'avete visto tramite proiezione stereografica



$[0, 2\pi] \times [0, \pi]$

$(0, 2\pi) \times (0, \pi)$

- IPERBOLOIDE IPERBOLICO (Quadrica rigata)



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$\boxed{\text{poniamo } a, b, c = 1 \text{ per semplicità}}$

$$x^2 + y^2 - z^2 = 1$$

- se $z > 0$ $\quad z = \sqrt{1 - x^2 - y^2}$

$$\mathbb{R}^2 \setminus \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 1 - x^2 - y^2 \leq 0 \right\} \xrightarrow{\varphi_1} \mathbb{R}^3$$

$$x^2 + y^2 \geq 1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ \sqrt{1 - x^2 - y^2} \end{pmatrix}$$

- φ_1 induce una biezione con l'immagine
- φ_1 è differenziabile
- $\varphi_1^{-1} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \operatorname{im} \varphi_1 \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}$

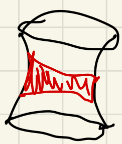
  è differenziabile

diffeomorfismo

φ_2 altra parametrizz. $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ -\sqrt{1-x^2-y^2} \end{pmatrix}$

servono altre param.

$\varphi_3, \varphi_4, \varphi_5, \varphi_6$ per parametrizzare



$$x^2+y^2-z^2=1$$

$$x=\sqrt{1-y^2+z^2}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \pm\sqrt{1-y^2+z^2} \\ y \\ z \end{pmatrix}$$

$$\mathbb{R}^2 \setminus \left\{ \begin{pmatrix} y \\ z \end{pmatrix} : \quad 1 - y^2 + z^2 \le 0 \right\}$$

$\begin{pmatrix} y \\ z \end{pmatrix}$

$$z^2 - y^2 \le 1$$