LEZIONE 10

CURVE (parametriche)

piane : $\alpha : J \longrightarrow \mathbb{R}^2$ ($J \subseteq \mathbb{R}$ intervallo) t.c. $\underbrace{\|\alpha'(t)\| = 1}_{\text{velocità unitaria}}$

$t \longmapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

di classe $\mathcal{C}^\infty$

$\det \begin{pmatrix} x' & -y' \\ y' & x' \end{pmatrix} = (x')^2 + (y')^2 = \|\alpha'\|^2 = 1$

Base di Frenet : $\begin{cases} \underline{t} = \alpha' = \begin{pmatrix} x' \\ y' \end{pmatrix} & \text{versore tangente} \\ \underline{n} = \text{vettore ortonormale ad } \alpha' \text{ t.c. base sia equiorientata} = \begin{pmatrix} -y' \\ x' \end{pmatrix} & \text{versore normale} \end{cases}$

Equazioni di Frenet : $\begin{cases} \underline{t}' = k \cdot \underline{n} \\ \underline{n}' = -k\,\underline{t} \end{cases} \rightsquigarrow \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} k x' \\ k y' \end{pmatrix}$

k : curvatura

$\rightsquigarrow \underline{t}' \cdot \underline{n} = k \cdot \underline{t} \cdot \underline{n} = k$

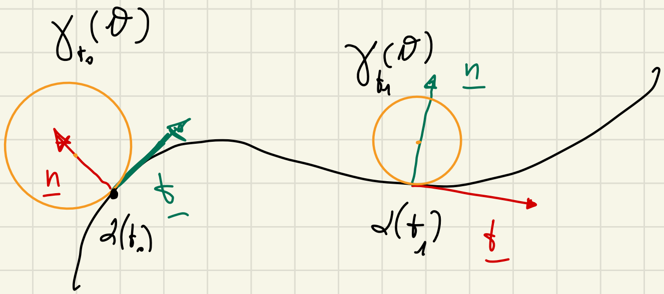
(·) $k(t_0) = 0 \rightsquigarrow t_0$ detto flesso

(·) $k(t_0) \neq 0 \rightsquigarrow \left| \dfrac{1}{k(t_0)} \right|$ raggio di curvatura (in t_0)

$$c(t_0) = \alpha(t_0) + \frac{1}{k(t_0)}\, \underline{n}(t_0)$$ centro di curvatura (in t_0)

$$\gamma_{t_0}(\theta) = c(t_0) + \frac{1}{k(t_0)}(\cos\theta)\cdot \underline{t} + \frac{1}{k(t_0)}(\text{sen}\,\theta)\,\underline{n}$$

cerchio osculatore in t_0



$$(x'^2 + y'^2)' = 2x'\cdot x'' + 2y'y''$$

nello spazio:

$$\alpha : J \to \mathbb{R}^3 \qquad \|\alpha'(t)\| = 1$$

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad \text{di classe } C^\infty$$

Base di Frenet:

$$\begin{cases} \underline{t} = \alpha' & \text{versore tangente} \\ \underline{n} = \dfrac{\alpha''}{\|\alpha''\|} & \text{versore normale} \\ \underline{b} = \underline{t} \times \underline{n} & \text{versore binormale} \end{cases}$$

Oss Questo perché se $1 = \|\alpha'\|$

$$\Rightarrow (\|\alpha'\|^2)' = 2\alpha'\cdot\alpha'' = 0 \Rightarrow \alpha' \perp \alpha''$$

Equazioni di Frenet :
$$\begin{cases} \underline{t}' = k\,\underline{n} \\ \underline{n}' = -k\,\underline{t} + \tau\,\underline{b} \\ \underline{b}' = -\tau\,\underline{n} \end{cases}$$

k : curvatura

τ : torsione

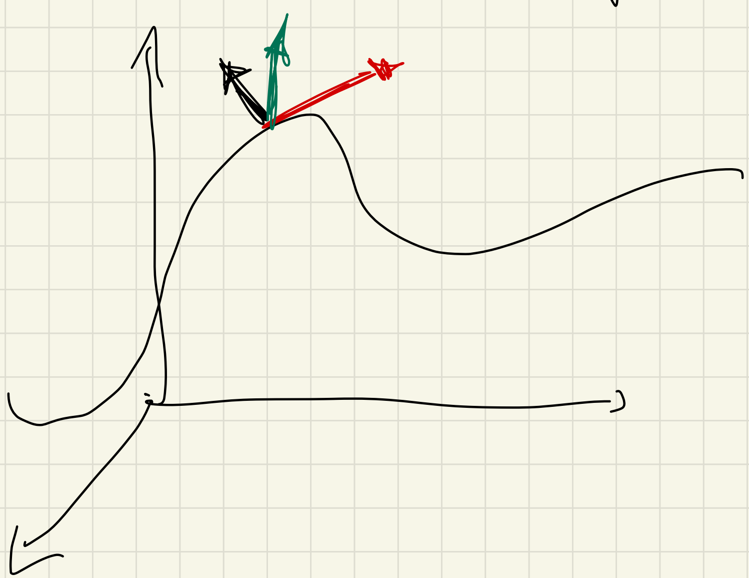
1° piano osculatore $\langle \underline{t}, \underline{n} \rangle$

piano rettificante $\langle \underline{t}, \underline{b} \rangle$

piano normale osculatore $\langle \underline{n}, \underline{b} \rangle$

piano osculatore affine $\alpha(t) + \langle \underline{t}, \underline{n} \rangle$

cerchio osculatore affine stesse formule come sopra : contenuto nel piano osculatore affine

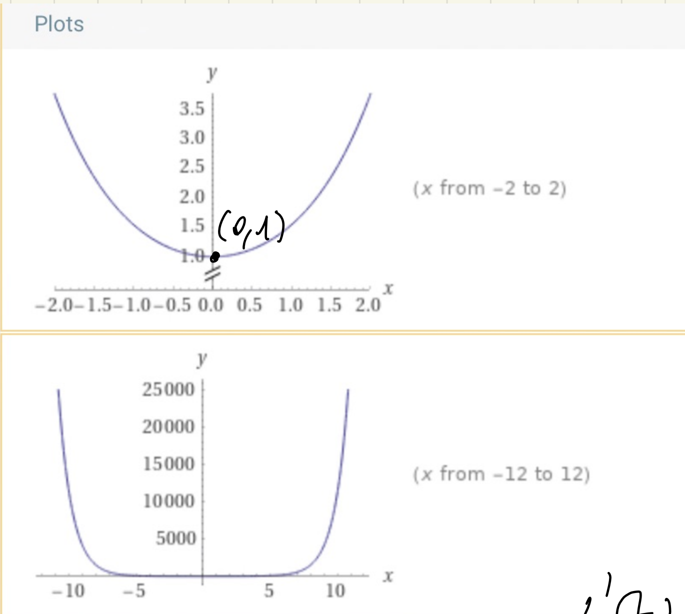


CATENARIA

$$\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \begin{pmatrix} t \\ \cosh(t) \end{pmatrix}$$

$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$

$$\operatorname{senh}(t) = \frac{e^t - e^{-t}}{2}$$



$$\cosh^2(t) - \operatorname{senh}^2(t) = 1$$

$$\alpha'(t) = \begin{pmatrix} 1 \\ \operatorname{senh}(t) \end{pmatrix}$$

$$\|\alpha'(t)\| = \sqrt{1^2 + \operatorname{senh}^2(t)}$$
$$= \sqrt{1 + \operatorname{senh}^2(t)}$$
$$= |\cosh(t)| \neq 1$$

$t = t(s)$ in modo tale che $\|\beta'\| = 1$

$$\left[\beta = \alpha(t(s)) : \delta' \longrightarrow \mathbb{R}^2 \atop s \longmapsto \beta(s) \right] ?$$

$$s(t) = \int_{t_0}^{t} \|\alpha'(u)\| \, du$$ lunghezza d'arco

$$t(s) = (s(t))^{-1}$$

$$s(t) = \int_{t_0}^{t} \|\alpha'\| \, du = \int_{t_0}^{t} |\cosh(u)| \, du =$$

$$= \int_{t_0}^{t} \cosh(u) \, du =$$

$$= \operatorname{senh}(t) - \operatorname{senh}(t_0)$$

$$t_0 = 0 \leadsto = \operatorname{senh}(t) = \frac{e^t - e^{-t}}{2}$$

$$f(s) = \operatorname{arcsenh}(t)$$

$$\overset{!}{=} \log\left(s + \sqrt{s^2+1} \right)$$

$$s = \frac{e^t - e^{-t}}{2} = \frac{e^t - 1/e^t}{2} =$$

$$= \frac{e^{2t} - 1}{2e^t}$$

$$2e^t \cdot s = e^{2t} - 1$$

$$e^{2t} - 2s\,e^t - 1 = 0$$

$$e^t = \frac{2s \overset{+}{\cancel{-}} \sqrt{4s^2+4}}{2} = s + \sqrt{s^2+1}$$

$$t = \log\left(s + \sqrt{s^2+1} \right)$$

$$\beta = \alpha(t(s)) : J' \longrightarrow \mathbb{R}^2$$

$$s \longmapsto \begin{pmatrix} \log(s+\sqrt{1+s^2}) \\ \cdot \end{pmatrix}$$

$$\cosh\left(\log\left(s+\sqrt{s^2+1}\right) \right) =$$

$$= \frac{\cancel{e}^{\cancel{\log}(s+\sqrt{s^2+1})} + \left(\cancel{e}^{\cancel{\log}(s+\sqrt{s^2+1})}\right)^{-1}}{2} =$$

$$= \frac{1}{2}\left(s + \sqrt{s^2+1} + \frac{1}{s+\sqrt{s^2+1}} \right) =$$

$$= \frac{1}{2}\left(\frac{s^2 + s^2+1 + 2s\sqrt{s^2+1} + 1}{s+\sqrt{s^2+1}} \right)$$

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2} s^2 + \cancel{2} s\sqrt{s^2+1} + \cancel{2}^{1}}{s+\sqrt{s^2+1}} =$$

$$= \frac{\left(s^2+1 + s\sqrt{s^2+1}\right)\left(s-\sqrt{s^2+1}\right)}{\cancel{s^2} - \cancel{s^2} - 1} =$$

$$= -\left(\cancel{s^3} + \cancel{s} + \cancel{s^2\sqrt{s^2+1}} - \cancel{s^2\sqrt{s^2+1}} + \right.$$

$$\left. - \sqrt{s^2+1} - \cancel{s^3} - \cancel{s}\right)$$

$$= +\sqrt{s^2+1}$$

Per la richiesta sono per lo stato vedere che

(1) $\tilde{\sigma} = \begin{pmatrix} \log\left(s+\sqrt{1+s^2}\right) \\ \sqrt{1+s^2} \end{pmatrix}$

è tale che $\|\tilde{\sigma}'\| = 1$

(2) è riparametrizzazione ovvero che se

$t = \log\left(s+\sqrt{1+s^2}\right) \Rightarrow \sqrt{1+s^2} = \cosh(t)$

Base di Frénet

$$\boxed{\underline{t} = \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{1+s^2} \\ s/\sqrt{1+s^2} \end{pmatrix}}$$

$$x'(s) = \frac{1 + \frac{1}{\cancel{2}} \frac{\cancel{2}s}{\sqrt{1+s^2}}}{s + \sqrt{1+s^2}} =$$

$$= \frac{\cancel{\sqrt{1+s^2} + s}}{\left(\sqrt{1+s^2}\right)\cancel{\left(s + \sqrt{1+s^2}\right)}}$$

$$= \frac{1}{\sqrt{1+s^2}}$$

$$y'(s) = \frac{s}{\sqrt{1+s^2}}$$

$$\boxed{\underline{n} = \begin{pmatrix} -y'(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} -s/\sqrt{1+s^2} \\ 1/\sqrt{1+s^2} \end{pmatrix}}$$

$$\begin{cases} \underline{t}' = \kappa \cdot \underline{n} \\ \underline{n}' = -\kappa \cdot \underline{t} \end{cases}$$

$$\underline{t}' = \beta'' = \begin{pmatrix} \dfrac{0 - \dfrac{s}{\sqrt{1-s^2}}}{(1-s^2)} \\ \\ \cdot \end{pmatrix} = \begin{pmatrix} \dfrac{-s}{(1-s^2)\sqrt{1-s^2}} \\ \\ \dfrac{1}{(1+s^2)\sqrt{1+s^2}} \end{pmatrix}$$

$$\frac{\sqrt{1+s^2} - s^2/\sqrt{1+s^2}}{(1+s^2)} = \frac{1+\cancel{s^2} - \cancel{s^2}}{(1+s^2)\sqrt{1+s^2}}$$

$$\kappa(s) = \frac{1}{s^2+1} \qquad r(s) = |s^2+1| = \\ = s^2+1$$

Cerchio osculatore

centro osculatore $\quad C(s) = \beta(s) + \frac{1}{k(s)} \underline{n}(s)$

$$= \begin{pmatrix} \log\left(s+\sqrt{1+s^2}\right) - \frac{\cancel{(s^2+1)}^{\sqrt{1+s^2}}\, s}{\cancel{\sqrt{1+s^2}}} \\ \sqrt{1+s^2} + \frac{\cancel{(s^2+1)}\sqrt{1+s^2}}{\cancel{\sqrt{s^2+1}}} \end{pmatrix} =$$

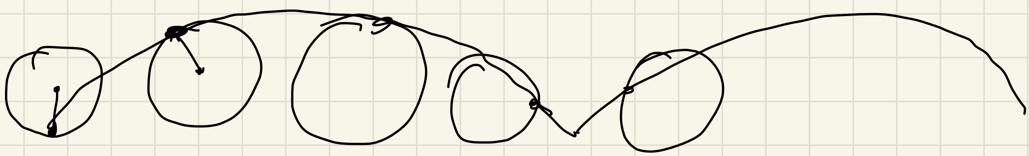
$$= \begin{pmatrix} \log\left(s+\sqrt{1+s^2}\right) + \sqrt{1+s^2} \\ 2\sqrt{1+s^2} \end{pmatrix}$$

raggio di curvatura $\quad R(s) = s^2+1$

CICLOIDE

Curva descritta da un punto su di una circonferenza che rotola senza strisciare su una retta

https://youtu.be/Q-4bHIH9vOI



Usiamo come parametro l'angolo θ:

$$\begin{cases} x(\theta) = R(\theta - \operatorname{sen}\theta) \\ y(\theta) = R(1 - \cos\theta) \end{cases} \qquad \theta \in (0, 2\pi)$$

Fissiamo $R = 1$ per semplicità.

$$\alpha = \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} \qquad \alpha'(\theta) = \begin{pmatrix} 1 - \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}$$

$$\|\gamma'\| = \sqrt{1 + \underline{\cos^2\vartheta} - 2\cos\vartheta + \underline{\operatorname{sen}^2\vartheta}}$$

$$= \sqrt{2 - 2\cos\vartheta}$$

$$= \sqrt{2}\sqrt{1-\cos\vartheta}$$

$$\boxed{\operatorname{sen}\frac{\vartheta}{2} = \pm\sqrt{\frac{1-\cos\vartheta}{2}}}$$

$$s(\vartheta) = \int_0^{\vartheta} \sqrt{2}\sqrt{1-\cos u}\, du$$

$$= \int_0^{\vartheta} \sqrt{2}\sqrt{2}\left|\operatorname{sen}\frac{u}{2}\right| du$$

$\vartheta \in [0, 2\pi]$

$\frac{\vartheta}{2} \in [0, \pi]$

$$= 2\cdot 2\int_0^{\vartheta} \frac{1}{2}\operatorname{sen}\left(\frac{u}{2}\right) du$$

$$= 4\left[-\cos\frac{u}{2}\right]_0^{\vartheta} = 4\left(-\cos\frac{\vartheta}{2} + 1\right)$$

$$= 4\left(1 - \cos\left(\frac{\vartheta}{2}\right)\right)$$

$$\theta(s) = 2 \arccos\left(\frac{s}{4} - 1\right)$$

$$\beta(s) = \begin{pmatrix} 2 \arccos\left(\frac{s}{4} - 1\right) - \operatorname{sen}\left(2 \arccos\left(\frac{s}{4} - 1\right)\right) \\ 1 - \cos\left(2 \arccos\left(\frac{s}{4} - 1\right)\right) \end{pmatrix}$$

Conti ?