ADVANCED STOCHASTIC PROCESSES - 7° LECTURE

From the last lecture, recall:

Theorem 3: If X is irreducible and recurrent, then, $\forall x \in S$,

$$\mu_x(y) := \mathbb{E}_x(N_{\tau_x}(y)), \quad \forall y \in S$$

are invariant measures, and they are s.t. $\mu_x(y) > 0 \quad \forall x, y \in S$.
Moreover, invariant measures only differ from a multiplicative factor.

As a consequence, we get:

Theorem 4: If X is irreducible and positive recurrent, then exists a unique invariant distribution π, s.t.

$$\pi(x) = \frac{1}{\mathbb{E}_x(\tau_x)}, \quad \forall x \in S$$

__Proof__ (Theorem 3)

First note that $\mu_x(y) = \mathbb{E}_x\left(\sum_{m=0}^{\tau_x - 1} \mathbb{1}_{\{X_m = y\}}\right) = \mathbb{E}_x\left(\sum_{m=1}^{\tau_x} \mathbb{1}_{\{X_m = y\}}\right)$

$$\left(\begin{array}{l} \text{if } x \neq y: \ \mathbb{1}_{\{X_0 = y\}} = 0, \ \mathbb{1}_{\{X_{\tau_x} = y\}} = 0 \\ \text{if } x = y: \ \mathbb{1}_{\{X_m = y\}} = 0 \ \forall m \neq 0, \tau_x \end{array}\right)$$

and that $\mu_x(x) = 1$.

- To show that μ_x is an invariant measure, we write

$$\mu_x(y) = \mathbb{E}_x\left[\sum_{m \geq 1} \mathbb{1}_{\{X_m = y\}} \mathbb{1}_{\{\tau_x \geq m\}}\right] = \sum_{m \geq 1} {}^{x}P^{(m)}_{x,y}$$

where ${}^{x}P^{(m)}_{x,y} := \mathbb{P}_x(X_1 \neq x, \dots, X_{m-1} \neq x, X_m = y) \quad \forall m \geq 1, \ x, y \in S$

Note that:

$$\begin{cases} {}^{x}P^{(1)}_{x,y} = \mathbb{P}_x(X_1 = y) = P_{x,y} \\ {}^{x}P^{(m)}_{x,y} = \sum_{z \neq x} {}^{x}P^{(m-1)}_{x,z} \cdot P_{z,y}, \quad \forall m \geq 2 \end{cases}$$

Then

$$\mu_x(y) = \sum {}^{x}P^{(m)}_{x,y} = P_{x,y} + \sum \sum {}^{x}P^{(m-1)} \cdot P = P_{x,y} + \sum \mu_x(z) P$$

$$\mu_x(y) = \sum_{m\geq 1} {}^xP^{(m)}_{x,y} = P_{xy} + \sum_{z\neq x}\underbrace{\sum_{m\geq 2} {}^xP^{(m-1)}_{x,z}}_{=\mu_x(z)} \cdot P_{z,y} = P_{xy} + \sum_{z\neq x} \mu_x(z) P_{z,y}$$

(since $\mu_x(x)=1$) $= \sum_{z\in S} \mu_x(z) P_{z,y} \implies \mu = \mu P$ ✓

- From the invariance property, we get

$\forall\, m\geq 1: \quad \mu_x = \mu_x P^m \iff$

$$\mu_x(y) = \sum_{z\in S} \mu_x(z) P^{(m)}_{z,y} = P^{(m)}_{xy} + \sum_{z\neq x} \mu_x(z) P^{(m)}_{z,y} \qquad \forall\, y\in S$$

* If $\mu_x(y)=0$ for some $y \implies P^{(m)}_{xy}=0 \ \forall\, m\in\mathbb{N}$ which contradicts irreducibility $\implies$ $\boxed{\mu_x(y)>0 \ \forall\, y\in S}$.

* If $\mu_x(y)=+\infty$ for some y, the identity $1 = \mu_x(x) = \sum_{z\in S} \mu_x(z) P^{(m)}_{z,x}$ would imply $P^{(m)}_{y,x}=0 \ \forall\, m\in\mathbb{N}$ which again contradicts irreducibility $\implies$ $\boxed{\mu_x(y) < +\infty, \ \forall\, y\in S}$

- Proof of uniqueness up to multiplicative factor:

Let λ an invariant measure for P, and w.l.o.g., let us assume that for a given $x\in S$, $\lambda(x)=1$ $(=\mu_x(x)) \to$ if not, we can normalize.

- First we show that $\lambda(y) \geq \mu_x(y) \ \forall\, y\in S$.

Using iteratively the invariance of λ, we get, $\forall\, y\in S$

$$\lambda(y) = \lambda P(y) = \sum_{z_1\in S} \lambda(z_1) P_{z_1,y} = \sum_{z_1\neq x} \lambda(z_1) P_{z_1,y} + P_{x,y} \quad \text{(since } \lambda(x)=1\text{)}$$

(iterate over $\lambda(z_1)$)
$$= \sum_{z_2\neq x} \lambda(z_2) \underbrace{\sum_{z_1\neq x} P_{z_2 z_1} P_{z_1,y}}_{{}^xP^{(2)}_{z_2,y}} + \underbrace{P_{x,y}}_{{}^xP^{(1)}_{x,y}} + \underbrace{\sum_{z_1\neq x} P_{x,z_1} P_{z_1,y}}_{{}^xP^{(2)}_{x,y}}$$

(iterate m times)
$$= \sum_{z_m\neq x} \lambda(z_m)\, {}^xP^{(m)}_{z_m,y} + \left(\sum_{k=1}^{m} {}^xP^{(k)}_{x,y}\right) \geq \sum^{m} {}^xP^{(k)}_{xy}$$

iterate n times $= \sum_{z_n \neq x} \lambda(z_n) \, {}^xp^{(n)}_{z_n,y} + \left(\sum_{k=1}^{n} {}^xp^{(k)}_{x,y} \right) \geq \sum_{k=1}^{n} {}^xp^{(k)}_{x,y}$ (first sum ≥ 0)

Letting $n \to \infty$, we get $\boxed{\lambda(y) \geq \sum_{k=1}^{\infty} {}^xp^{(k)}_{x,y} = \mu_x(y)}$

- We finally show that $\lambda(y) = \mu_x(y) \ \forall y \in S$.

Let $\pi := \lambda - \mu_x \geq 0$, which is also invariant measure, with $\pi(x)=0$.

$$0 = \pi(x) = \sum_{z \in S} \pi(z) P^{(n)}_{z,x} \geq \pi(y) P^{(n)}_{y,x} \implies \pi(y) = 0 \iff \boxed{\lambda = \mu_x}$$

($\forall y \in S, \forall n$; $P^{(n)}_{y,x} > 0$ from irreducibility, and suitable n)

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We conclude with the general Theorem on convergence to stationarity.

Theorem 5 [Convergence to stationarity]
Let X be an irreducible, aperiodic and positive recurrent MC on S.
Then, $\forall \mu \in \mathcal{P}(S)$, it holds $\|\mu P^n - \pi\|_{TV} \xrightarrow[n\to\infty]{} 0$
and equivalently $\sum_{x \in S} |P^{(n)}_{x,y} - \pi(y)| \xrightarrow[n\to\infty]{} 0 \quad \forall y \in S$

Proof: Let X be a MC with $X_0 \sim \mu$ and Y a MC with $Y_0 \sim \pi$ (so that, $\forall n \in \mathbb{N}$, $Y_n \sim \pi$).

As we have done for finite S, we consider a coupling that let the MCs move independently up to time $\tau = \inf_{n \geq 0} \{X_n \neq Y_n\}$ and then move together. Then, by the Prop. [Coupling and TV-distance],

$$\|\mu P^n - \pi\|_{TV} = \|\mu P^n - \pi P^n\|_{TV} \leq Q(X_n \neq Y_n) = Q(\tau > n)$$

and to conclude we are left to verify that

$$Q(\tau > n) \xrightarrow[n\to\infty]{} 0 \iff Q(\tau < \infty) = 1$$

To show this we can take the coupling $\tilde{Q} = P_\mu \otimes P_\pi$ namely

$Q(\tau > n) \xrightarrow[n\to\infty]{} 0 \iff Q(\tau < \infty) = 1$

To show this we consider the coupling $\tilde{Q} = \mathbb{P}_\mu \otimes \mathbb{P}_\pi$, namely we let $Z_n := (X_n, Y_n)$, $\forall n \geq 0$, the MC on S^2 s.t. $Z_0 \sim \mu \times \pi$ and $\tilde{q}_{(x,x')|(y,y')} = P_{x,y} \cdot P_{x'y'}$ (so that $(X_n) \perp (Y_n)$)

and denote by $\tilde{Q}$ it's law. Then we have:

* Z is irreducible since P is irreducible and aperiodic
  (verify! Without aperiodicity, Z may not be irreducible)
* Z has the invariant distribution $\tilde{\pi} \in \mathcal{P}(S^2)$,
  with $\tilde{\pi}(x,y) = \pi(x)\pi(y)$ (verify)

From Theorems 2, 3, 4, it turns out that Z is positive recurrent (with $\mathbb{E}_{(x,y)}(\tau_{(x,y)}) = \mathbb{E}_x(\tau_x)\mathbb{E}_y(\tau_y) < \infty$)
and in particular $\tilde{Q}(\tau_{(x,y)} < \infty) = 1 \quad \forall (x,y) \in S^2$

At last, notice that $Q(\tau < \infty) = \tilde{Q}(\tau < \infty)$, and
$\tilde{Q}(\tau < \infty) = \tilde{Q}(\exists x \in S : \tau_{(x,x)} < \infty) = 1$ #

Exercise: Show that an irreducible MC $X = (X_n)_{n \geq 0}$ on a finite space S is positive recurrent.
