LEZIONE 1

07/03/22

mestelle@math.unipd.it

---

(1) $X = \{1, 2, 3\}$

$\tau = \{\emptyset, X, \{1,2\}, \{2,3\}, \{2\}\}$

é una topologia su X.

(i) $\emptyset, X \in \tau \Rightarrow$ VERA

(ii) $A \in \tau$ $\quad A \cup \emptyset = A \in \tau$

$A \cup X = X \in \tau$

$\{1,2\} \cup \{2,3\} = X \in \tau$

$\{1,2\} \cup \{2\} = \{1,2\} \in \tau$

$\{2,3\} \cup \{2\} = \{2,3\} \in \tau$

$\Rightarrow \bigcup$ 2 aperti è aperto.

$\Rightarrow \bigcup_{\text{finita}}$ aperti è aperto

$\not\Rightarrow \bigcup_{\text{arbitraria}}$ è aperto in generale

in questo caso basta $\bigcup$ finita

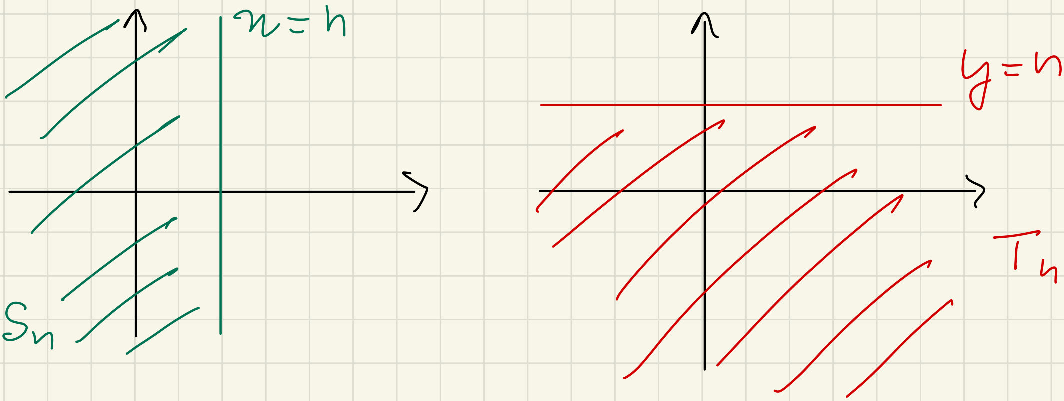
$\Rightarrow$ (ii) è VERA

(iii) VERA

③ $X = \mathbb{R}^2$

$$S_n = \{(x,y) \in X : x < n\}$$

$$T_n = \{(x,y) \in X : y < n\}$$



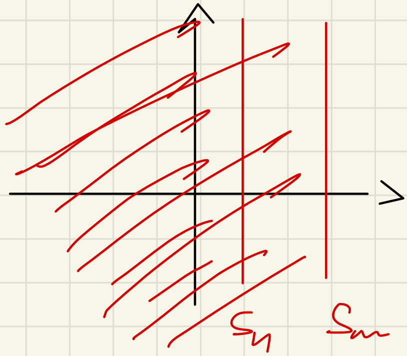
$$\tau_1 = \{\emptyset, X, S_n : n \in \mathbb{Z}\}$$

$$\tau_2 = \{\emptyset, X, T_n : n \in \mathbb{Z}\}$$

Mostrare che sono 2 topologie

Facciamolo per τ_1.

(i) $\emptyset, X \in \mathcal{T}_1$



(ii) $S_n \cup S_m =$

$$= S_{\max\{n,m\}}$$

$I \subseteq \mathbb{Z}$

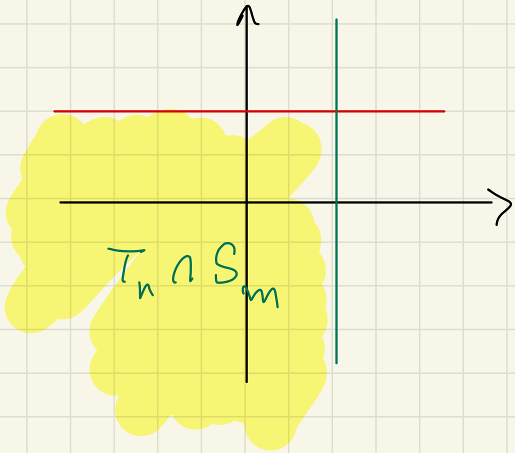
$$\bigcup_{i \in I} S_i = S_{\sup I} \in \mathcal{T}_1 \qquad S_\infty := X$$

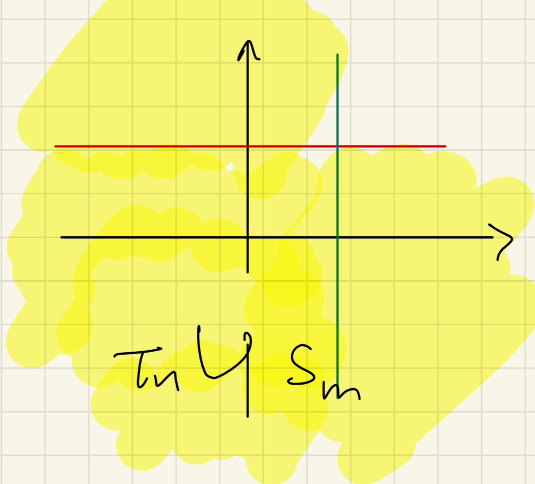
(iii) $S_n \cap S_m = S_{\min\{n,m\}} \in \mathcal{T}_1$

$\mathcal{T}_1 \cap \mathcal{T}_2 \overset{?}{=} \{\emptyset, X\}$ topologie banale

$\mathcal{T}_1 \cup \mathcal{T}_2 \overset{?}{=} \{\emptyset, X, S_n, T_n : n \in \mathbb{Z}\}$

é ancore una topologie ? NO





$\notin \tau_1 \cup \tau_2$ $\qquad\qquad$ $\notin \tau_1 \cup \tau_2$

Se $\tau_1' = \{\emptyset, X, S_n, n \in \mathbb{R}\}$

(i) ✓

(ii) es. $\bigcup_{n<2} S_n \overset{?}{=} S_2$ Sì

È ancora topologia!

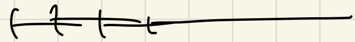
ES 5

$$\mathcal{B} = \{(a, +\infty) \quad a \in \mathbb{R}\}$$

1) è una base per una topologia

2) strettamente meno fine della Euclidea

- $$\bigcup_{B \in \mathcal{B}} B = \mathbb{R}$$

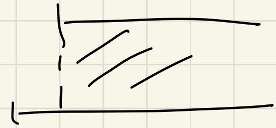


$$= \bigcup_{a \in \mathbb{R}} (a, +\infty) = \mathbb{R}$$

✓

$a, b \in \mathbb{R}$

- $$(a, +\infty) \cap (b, +\infty) = (\max\{a,b\}, +\infty)$$

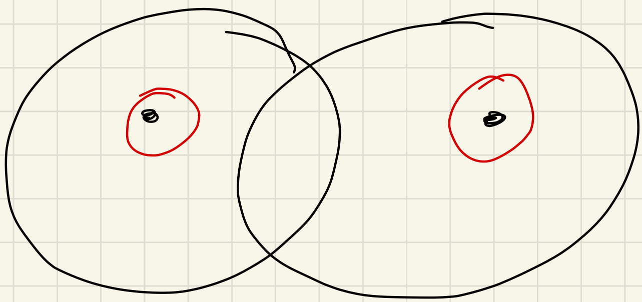


✓

Dunque, per il teorema dell'esistenza della topologia generata segue (1)

(2) $\tau_B \subsetneq \tau_{Euclidea}$

PIÚ FINE $\equiv$ PIÚ APERTI



Basta mostrare:

- Preso un elemento di base allora questo è aperto in $\tau_{Euclidea}$

  $(a, +\infty) \in \tau_{Euclidea}$

- $(0, 1) \in \tau_{Euclidea}$

  $(0, 1) \notin \tau_B$ perché τ_B contiene solo $\emptyset$, $\mathbb{R}$, semirette $(a, +\infty)$

(6) $\mathcal{T}_1, \mathcal{T}_2$ sono due topologie su X

$\mathcal{B}_1, \mathcal{B}_2$ basi

(P1)

- Se $\forall B \in \mathcal{B}_1 \quad \exists\, B_i \in \mathcal{B}_2 \quad \forall i \in I$ tale che $B = \bigcup_{i \in I} B_i$ allora

$\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Dim

$$U \in \mathcal{T}_1 \qquad U = \bigcup_{j \in J} B^{(j)} \qquad B^{(j)} \in \mathcal{B}_1$$

$$\forall j \in J : B^{(j)} = \bigcup_{i \in I_j} B_i^{(j)} \qquad B_i^{(j)} \in \mathcal{B}_2$$

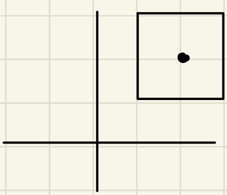
$$\Rightarrow U = \bigcup_{\substack{(i,j) \\ \cap \\ \bigcup_{j \in J} I_j \times \{j\}}} B_i^{(j)} \Rightarrow U \in \mathcal{T}_2 \qquad \square$$

In particolare se dimostro (P1) e (P2) $\Rightarrow \tau_1 = \tau_2$

In un esempio pratico: la topologia su $\mathbb{R}^2$ generata dai $\square$ è la topologia Euclidea (generata dai $\bigcirc$)

$$\mathcal{B} = \left\{ Q(x, r) : \begin{array}{l} x \in \mathbb{R}^2 \\ r > 0 \end{array} \right\} \qquad x = (x_1, x_2)$$

$$Q(x, r) = \left\{ (y_1, y_2) \in \mathbb{R}^2 : |x_1 - y_1|, |x_2 - y_2| < r \right\}$$



$\exists r$

lato $2r$

centro $x = (x_1, x_2)$

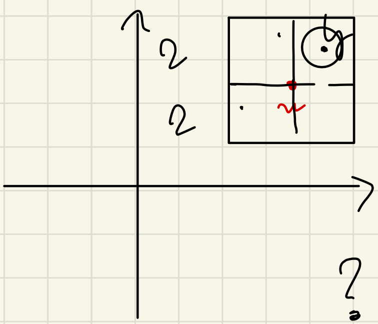
$$\mathcal{B}_{euclid} = \left\{ D(x, r) : x \in \mathbb{R}^2, r > 0 \right\}$$

$$D(x, r) = \{ y : \| y - x \| < r \}$$

In pratica devo mostrare che

$$\square = \bigcup \bigcirc$$

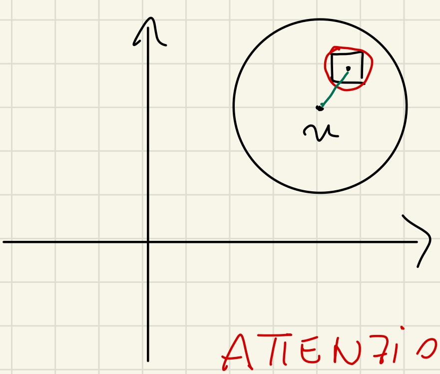
$$\bigcirc = \bigcup \square$$



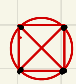
$$Q(x, r) = \bigcup_{y \in Q(x,r)} D(y, ?_1)$$

$$?_1 < r - \max\{|y_1 - x_1|, |y_2 - x_2|\}$$

$$D(x, r) = \bigcup_{y \in D(x,r)} Q(y, ?_2)$$



$?_2$ $\quad r - \|y - x\|$



ATTENZIONE SVISTA A LEZIONE

$$?_2 < \frac{r - \|y - x\|}{\cancel{2}\sqrt{2}}$$

Es prendiamo uno spazio topologico $(X, \mathcal{T})$, I insieme di indici

Chiamiamo "BOX TOPOLOGY" la topologia su $\prod_{i \in I} X$ avente

base $\mathcal{B} = \{ \prod_{i \in I} U_i \ : \ U_i \in \mathcal{T} \}$

• ha le proprietà di una base?

In particolare se $X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_{\text{Euclidea}}$

$$\left(\prod_{i \in I} \mathbb{R} \, , \, \mathcal{T}_{\mathcal{B}} \right)$$

$\# I = n \in \mathbb{N}$ $\qquad \mathbb{R}^n \rightsquigarrow$ topologia Euclidea

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rightsquigarrow$ box topology

$\# I = +\infty$???

LO VEDREMO IN FUTURO ...