6° LECTURE - ADVANCED STOCHASTIC PROCESSES

Def [Visits to a state - local time]: $\forall y \in S$, let

$$\forall m \geq 0: \quad N_m(y) := \sum_{n=0}^{m-1} \mathbb{1}_{\{X_n = y\}}, \quad N(y) := \sum_{n=0}^{+\infty} \mathbb{1}_{\{X_n = y\}}$$

$N_m(y)$ = # of visits to y, up to time m; $N(y)$ = # of visits to y

and set $$G(x,y) := \mathbb{E}_x(N(y)) = \sum_{n=0}^{+\infty} \mathbb{P}_x(X_n = y) = \sum_{n=0}^{+\infty} P_{x,y}^{(n)}$$

↳ called Green function: $G: S \times S \to \overline{\mathbb{R}}_+$

Theorem 1 [Criterion for recurrence]: $\forall x, y \in S$, it holds

$$G(x,y) = \begin{cases} \dfrac{F(x,y)}{1 - F(y,y)} & \text{if } x \neq y \\[2ex] \dfrac{1}{1 - F(y,y)} & \text{if } x = y \end{cases}$$

Remark: Then y is recurrent $(F(y,y) = 1) \iff G(y,y) = +\infty$.
Convinient, as $G(y,y) = \sum_{n=0}^{\infty} P_{y,y}^{(n)}$, explicit sum.

The proof of the above theorem use the following Lemma

Lemma: $\forall y \in S$, define iteratively the k-th passage times in y

$$\tau_y^1 \equiv \tau_y, \quad \tau_y^k = \inf\{n > \tau_y^{k-1} : X_n = y\} \quad \forall k \geq 2$$

Then
$$\boxed{\mathbb{P}_x(\tau_y^k < \infty) = F(x,y) F(y,y)^{k-1}}$$

Proof [of Lemma]: By induction over k.

- for $k = 1$, obvious from def. of $F(x,y)$;
- for $k \geq 2$, by the strong Markov property:

• for $k \geq 2$, by the strong Markov property:

$$\mathbb{P}_x\left(\tau_y^k < \infty\right) = \mathbb{E}_x\left(\mathbb{P}_x\left(\tau_y^k < \infty \,\middle|\, \mathcal{F}_{\tau_y^{k-1}}\right) \cdot \mathbb{1}_{\{\tau_y^{k-1} < \infty\}}\right)$$

$X_{\tau_y^{k-1}} = y$

$$= \mathbb{E}_x\left(\mathbb{P}_y\left(\tau_y < \infty\right) \mathbb{1}_{\{\tau_y^{k-1} < \infty\}}\right) = F(y,y)\,\mathbb{P}_x\left(\tau_y^{k-1} < \infty\right)$$

inductive hypothesis

$$\overset{\text{inductive hypothesis}}{=} F(y,y)F(y,y)^{k-2} \cdot F(x,y)$$

#

Proof [of Theorem - criterion for recurrence] :

Note that $N(y) = \mathbb{1}_{\{X_0 = y\}} + \sum_{k=1}^{\infty} \mathbb{1}_{\{\tau_y^k < \infty\}}$. Averaging w.r.t. $\mathbb{P}_x$

$$G(x,y) = \mathbb{E}_x(N(y)) = \delta_x(y) + \sum_{k=1}^{\infty} \mathbb{P}_x\left(\tau_y^k < \infty\right)$$

(from Lemma) $= \delta_x(y) + \sum_{k=1}^{+\infty} F(x,y)F(y,y)^{k-1} = \delta_x(y) + \dfrac{F(x,y)}{1 - F(y,y)}$ #

<u>Consequences</u>: If X is irreducible and $x \in S$ is recurrent, then:

1. all states $y \in S$ are recurrent (recurrence is a property of X)

<u>Proof</u>: We know that $G(x,x) = +\infty$.

For $y \in S$, let $l, k \in \mathbb{N}$ s.t. $p_{x,y}^{(k)} > 0$, $p_{y,x}^{(l)} > 0$ (irreduc.) so that

$$p_{y,y}^{(l+m+k)} \geq p_{y,x}^{(l)} \cdot p_{x,x}^{(m)} \cdot p_{x,y}^{k} \longrightarrow$$

. Then

$$G(y,y) = \sum_{m \geq 0} p_{y,y}^{(m)} \geq \sum_{m \geq 0} p_{y,y}^{(l+m+k)} \geq \underbrace{p_{y,x}^{(l)}}_{>0}\,\underbrace{p_{x,y}^{(k)}}_{>0} \underbrace{\sum_{m \geq 0} p_{x,x}^{(m)}}_{G(x,x) = +\infty} = +\infty$$ #

2. $F(x,y) = F(y,x) = 1 \quad \forall x,y \in S$

<u>Proof</u>: $0 = \mathbb{P}_x(\tau_x = \infty) \underset{\forall k \in \mathbb{N}}{\geq} \mathbb{P}_x(\tau_x = \infty, X_k = y) \underset{\text{Markov prop.}}{=} \mathbb{P}_x(X_k = y)\,\mathbb{P}_y(\tau_x = \infty)$

$$= \underbrace{p_{x,y}^{(k)}}_{>0} \left(1 - F(y,x)\right) \Longleftrightarrow F(y,x) = 1$$

for some $k \in \mathbb{N}$ due to irreduc.

(changing x and y we get similarly $F(x,y) = 1$. //

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Changing x and y, we get similarly $F(x,y)=1$. #

Corollary: If S is finite and X is irreducible then X is recurrent

Proof: $\forall x\in S$: $\sum_{y\in S} G(x,y) = \sum_{n=0}^{+\infty} \underbrace{\sum_{y\in S} P_{xy}^{(n)}}_{1} = +\infty$

But since S is finite, this can be true only if $\exists\, y\in S$ s.t. $G(x,y)=\infty$. From the recurrence criterion, as $F(x,y)>0$ from irr.,

$$G(x,y) = \frac{F(x,y)}{1-F(y,y)} = \infty \iff F(y,y)=1,$$

hence y (and all states in S) are recurrent. #

Existence of invariant measures and distribution

Theorem 2: If X is irreducible on S and transient $\Longrightarrow$ no invariant distribution

Proof: $G(x,y) = \sum_{n\geq 0} P_{x,y}^{(n)} < \infty$ $\forall x,y\in S$ (from transience and thm 1)

Then $P_{x,y}^{(n)} \xrightarrow[n\to\infty]{} 0$ and hence $\underset{\sum_{x\in S}\mu(x)P_{x,y}^{(n)}}{\mu P^n(y)} \to 0$, $\forall \mu\in\mathcal{P}(S)$, $\forall y\in S$

If μ would be an invariant distr., we would have $\mu P^n = \mu$, $\forall n\in\mathbb{N}$, and taking the limit $n\to\infty$: $\mu=0$, which is a contradiction. #

Remark: Under the above hypotheses, there may exist invariant measures

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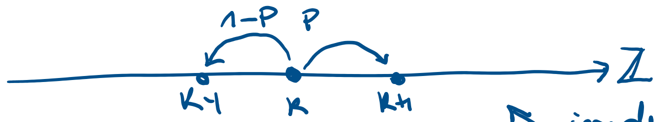


$$\mathbb{P}(S_n = j \mid S_{n-1} = k) = \begin{cases} p & \text{if } j = k+1 \\ 1-p & \text{if } j = k-1 \\ 0 & \text{otherwise} \end{cases}$$

↖ irreducible

If $p \neq \frac{1}{2}$, by Stirling approximation one gets

$$P_{0,0}^{(2n)} = \binom{2n}{n} p^n (1-p)^n \approx \frac{1}{\sqrt{\pi n}} \underbrace{[4p(1-p)]}_{<1}{}^n$$

Hence, $G(0,0) = \sum_{n\geq 0} P_{0,0}^{(n)} < \infty$, implying that (S_n) is transient

While, no invariant distribution exists, one can check that the following measures are invariant:

- $\mu_1(x) = 1 \quad \forall x \in \mathbb{Z}$
- $\mu_2(x) = \left(\frac{p}{1-p}\right)^x$

(Verify that, $\forall x \in \mathbb{Z}$
$p\,\mu(x-1) + (1-p)\,\mu(x+1) = \mu(x)$)

Theorem 3: If X is irreducible and recurrent, then, $\forall x \in S$,

$$\mu_x(y) := \mathbb{E}_x\left(N_{\tau_x}(y)\right), \quad \forall y \in S$$

are invariant measures, and they are s.t $\mu_x(y) > 0 \quad \forall x, y \in S$.

Moreover, invariant measures only differ from a multiplicative factor.

Remark:

1. Notice that the measure μ_x is not necessarily finite, hence the existence of an invariant distr. is not garanted.

Indeed:

$$\sum_{y\in S} \mu_x(y) = \mathbb{E}_x\left(\sum_{y\in S} N_{\tau_x}(y)\right) = \mathbb{E}_x(\tau_x)$$

- Under solely recurrence $\mathbb{E}_x(\tau_x)$ may be ∞ → no inv. distr

- Under solely recurrence, $\mathbb{E}_x(\tau_x)$ may be ∞ $\to$ no inv. distr
- Under positive recurrence of x : $\mathbb{E}_x(\tau_x) < \infty$, hence:

$$\pi_x(y) := \frac{\mu_x(y)}{\mathbb{E}_x(\tau_x)}, \quad y \in S \quad \text{is } \mathcal{P}(S)$$

and is s.t. $\quad \pi_x(x) = \dfrac{1}{\mathbb{E}_x(\tau_x)}$

But since μ_x is unique up to multiplicative constant, i.e. $\mu_x = c\,\mu_y$ for some $c = c(x,y) > 0$, $\forall\, x, y \in S$, it turns out

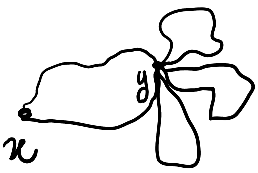
- $\mathbb{E}_y(\tau_y) < \infty \quad \forall\, y \in S$ (positive recurrence is a property of the MC)

- $\forall\, x, y \in S$: $\pi_x = \pi_y = \pi$ and then

$$\pi_y(y) = \frac{1}{\mathbb{E}_y(\tau_y)} = \frac{\mu_x(y)}{\mathbb{E}_x(\tau_x)} = \pi_x(y) \iff \mathbb{E}_x(\tau_x) = \mu_x(y)\,\mathbb{E}_y(\tau_y)$$

$$\iff \mathbb{E}_x(\tau_x) = \mathbb{E}_x(N_{\tau_x}(y))\,\mathbb{E}_y(\tau_y)$$

$\llcorner$ idea $\to$



- 4 passage to y before τ_x
- all of them take a time $\mathbb{E}_y(\tau_y)$

As a consequence of theorem 3 and of the Remark that followed it, we get the proof of the following result:

Theorem 4: If X is irreducible and positive recurrent, then exists a unique invariant distribution π, s.t.

$$\pi(x) = \frac{1}{\mathbb{E}_x(\tau_x)}, \quad \forall\, x \in S$$

Remark: This implies that if S is finite and X is irreducible,
Then the MC is positive recurrent (put things together).