## Recall:

**Theorem** [Convergence to stationarity – $S$ finite]

Let $X = (X_n)_{n \in \mathbb{N}_0}$ be a strongly irreducible MC on $S$ finite. Then

* it exists a unique invariant distribution $\pi \in \mathcal{P}(S)$
* $\pi$ is s.t. $\pi(x) > 0 \ \forall x \in S$
* it exists $\gamma = \gamma(m, \varepsilon) \in (0,1)$ s.t. , $\forall \mu \in \mathcal{P}(S)$:

related to condition of strong irreducibility

(2) $$\boxed{\| \mu P^n - \pi \|_{TV} \leq \gamma^n}$$

**Proof**

1: We first prove that $\exists \gamma \in (0,1)$ s.t. , $\forall y, y' \in S$

(3) $$\boxed{\| \delta_y P^n - \delta_{y'} P^n \|_{TV} \leq \gamma^n}$$

2: We then show that (3) implies the statements.

**Step 1**: Let $X$ the MC with $X_0 = y$, and $Y$ the MC with $Y_0 = y'$, both evolving with transition matrix $P$ (and law $\mathbb{P}_y, \mathbb{P}_{y'}$ respect.).

Consider the coupling $Q_{y,y'}$ of $\mathbb{P}_y$ and $\mathbb{P}_{y'}$ s.t. $X$ and $Y$ move independently as soon as they reach a common value, and then move together. Formally, we define joint trans. prob.

$$q_{(y,y'),(x,x')} = \begin{cases} P_{y,x} \cdot P_{y',x'} & \text{if } y \neq y' \\ P_{y,x} \cdot \delta_x(x') & \text{if } y = y' \end{cases}$$

Notice that $q : (S \times S) \times (S \times S) \longrightarrow [0,1]$ has marginals $P_{\cdot,\cdot}$:

$$\sum_{x' \in S} q_{(y,y')(x,x')} = P_{y,x} \quad \text{and} \quad \sum_{x \in S} q_{(y,y')(x,x')} = P_{y',x'}$$

Then $q_{(y,y'),(\cdot,\cdot)}$ is a coupling of $P_{y,\cdot}$ and $P_{y',\cdot}$, corresponding

Then $q_{(y,y'),(\cdot,\cdot)}$ is a coupling of $P_{y,\cdot}$ and $P_{y',\cdot}$, corresponding to a coupling $Q_{y,y'}$ of $P_y$ and $P_{y'}$.

We use strong irreducibility to show $Q_{y,y'}(X_m = Y_m) > \varepsilon$, $\forall y,y' \in S$

$\hookrightarrow m, \varepsilon$ s.t. $P^{(m)}_{x,y} > \varepsilon$ $\forall x,y \in S$

Let $\tau = \inf\{n \geq 0 : X_n = Y_n\}$. Then

$$Q_{y,y'}(\tau \leq m) = \sum_{x \in S} Q_{y,y'}(X_m = Y_m = x) = \sum_{x \in S} q^{(m)}_{(y,y'),(x,x)}$$

from definition: $q_{(y,y')(x,x)} \geq P_{y,x} \cdot P_{y',x}$

$$\geq \sum_{x \in S} P^{(m)}_{y,x} \cdot P^{(m)}_{y',x} \geq \varepsilon$$

$\hookrightarrow$ strong irreducibility

From the Markov property:

$$Q_{y,y'}(\tau > (k+1)m) = \mathbb{E}_{Q_{y,y'}}\left(Q_{y,y'}(\tau > (k+1)m \mid (X_{km}, Y_{km}))\right)$$

$$= \mathbb{E}_{Q_{y,y'}}\left(Q_{X_{km},Y_{km}}(\tau > m) \cdot \mathbb{1}_{\{X_{km} \neq Y_{km}\}}\right)$$

$$\leq \sup_{x,x'} Q_{x,x'}(\tau > m) \cdot Q_{y,y'}(X_{km} \neq Y_{km})$$

$$\leq (1-\varepsilon) Q_{y,y'}(\tau > k \cdot m) \leq \ldots \text{iterating} \leq (1-\varepsilon)^{k+1}$$

From the Proposition (coupling and TV-distance)

$$\|\delta_y P^n - \delta_{y'} P^n\|_{TV} \leq Q_{y,y'}(X_n \neq Y_n) = Q_{y,y'}(\tau > n) \leq (1-\varepsilon)^{\lfloor \frac{n}{m} \rfloor} =: \gamma^n$$

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Step 2: Let $\underline{\pi}_n(x) = \inf_y P^{(n)}_{y,x}$ and $\overline{\pi}_n(x) = \sup_y P^{(n)}_{y,x}$.

Then $\underline{\pi}_n(x) \nearrow_n$ and $\overline{\pi}_n(x) \searrow_n$ (verify) and then converge to the limit $\underline{\pi}(x)$ and $\overline{\pi}(x)$, respectively, $\forall x \in S$.

From (3) (proved in step 1), it holds

$$\|\overline{\pi}_n - \underline{\pi}_n\|_{TV} \leq |S| \cdot \gamma^n \xrightarrow[n\to\infty]{} 0 \implies \underline{\pi} = \overline{\pi} =: \pi$$

which implies (2): 
$$\begin{cases} \mu P^n(x) = \sum_y \mu(y) P^{(n)}_{y,x} \leq \inf_y P^{(n)}_{y,x} = \underline{\pi}_n \\ \quad\text{''} \qquad\qquad \text{''} \qquad \geq \sup_y P^{(n)}_{y,x} = \overline{\pi}_n \end{cases}$$

As a consequence, we also get:

* Uniqueness of the invariant measure: $\nu P = \nu \iff \nu P^n = \nu \quad \forall n \in \mathbb{N}$
  and taking the limit $n \to \infty$ : $\nu = \pi$
* $\pi(x) > 0 \quad \forall x \in S$, as limits of strictly positive sequences.
* Since $\forall \mu \in \mathcal{P}(S)$ we can write $\mu P^n = \sum_{y \in S} \mu(y) \cdot \delta_y P^n$,
  by the triangular inequality
  $$\| \mu P^n - \pi \|_{TV} \leq \sum_{y \in S} \mu(y) \underbrace{\| \delta_y P^n - \pi \|_{TV}}_{\gamma^n} \leq \gamma^n$$

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## * COUNTABLE SPACE

Let $S$ be a general countable space (not necessarily finite), and $(X_n)_{n \geq 0} = X$ a MC evolving on $S$. To study the existence of invariant distr. for $X$, and the conditions which garantee convergence toward it of $\mathcal{L}_\mu(X_n)$, we need to introduce further properties.

### Transience and recurrence

**Def:** $\forall y \in S$, we define the first passage time in $y$
$$\tau_y := \inf \{ n > 0 : X_n = y \} \in \overline{\mathbb{R}}_+ \quad \text{(or first entrance in } y\text{)}$$
and $\forall x \in S$, set
$$F(x,y) := \mathbb{P}_x(\tau_y < \infty) = \mathbb{P}_x(\exists n \in \mathbb{N} : X_n = y)$$
probability of a a.s. finite passage from $x$ to $y$

F for "first entrance"

**Remark:** From the second identity, we have
$$F(x,y) = \mathbb{P}_x\Big(\bigcup_{n > 0} \{X_n = y\}\Big) \geq P^{(n)}_{x,y} \quad \forall n \in \mathbb{N}$$
$\to$ if $X$ is irreducible, then $F(x,y) > 0$.

Def [Recurrence and transience]: A state $x \in S$ is called:

- <u>recurrent</u> if $F(x,x) = 1$ ($\tau_x$ is $\mathbb{P}_x$-a.s finite)
  - * positive recurrent if $\mathbb{E}_x(\tau_x) < \infty$
  - * null recurrent if it is recurrent but $\mathbb{E}_x(\tau_x) = +\infty$
- <u>transient</u> if $F(x,x) < 1$ ($\mathbb{P}_x(\tau_x = +\infty) > 0$)

(the nature of a state will provide necessary conditions for convergence)