3° LECTURE - ADVANCED STOCHASTIC PROCESSES

Recall: Wright's model is a MC $(X_n)_{n\geq 0}$ s.t.

* X_n = # of individuals of type A in n-generation
  so that $X_n \in S = \{0, 1, \dots, N\}$ N = size of population, fixed
* Each new individual chose an ancestor u.a.r. (prob. $1/N$) and copy its type
  → $X_n \mid X_{n-1} \sim Bin\left(N, \frac{X_{n-1}}{N}\right)$
  → $P_{j,k} = \mathbb{P}(X_n = k \mid X_{n-1} = j) = \mathbb{P}\left(Bin\left(N, \frac{j}{N}\right) = k\right), \ \forall j, k \in S$

Remarks:

1. X is a MC on $\{0, \dots N\}$ s.t. $P_{j,k} > 0 \quad \forall j \in \{1, \dots, N-1\}, k \in S$
   while $P_{0,0} = 1$ and $P_{N,N} = 1$ (hence $P_{0,j} = 0 \ \forall j \neq 0$, $P_{N,j} = 0 \ \forall j \neq N$).
   When this happens, we call the state absorbing, as the MC that ends in it, stays there forever.
   Notice that states 0 and N correspond to the situation in which a type has dominated the other (0 → only type B, N → only type A)
   ↳ called fixation

2. The Wright's model can be equivalently defined as the evolution of the frequence of type A among the generations (this is indeed the most common representation):
   $\forall n \in \mathbb{N}_0: \quad Z_n := \frac{X_n}{N} \in \left\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\right\} = S'$

   Then (easy check) $(Z_n)_{n\geq 0}$ is a MC with trans. matrix
   $$P_{x,y} = \mathbb{P}(Bin(N, x) = N \cdot y), \quad \text{for } x, y \in S'$$
   This representation is convenient because it allows to consider

This representation is convenient because it allows to consider large value of N (hence the limit $N \to \infty$) keeping S' compact.

We state a general result of martingale theory, which is powerful and that will be applied in our context to compute some relevant quantities.

<u>Theorem</u> (Martingale convergence theorem)

Let $(X_n)_{n \geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ - martingale

- If $(X_n)_{n\geq 0}$ is L^1 - bounded (i.e. $\mathbb{E}(|X_n|) \leq c$, $\forall n \geq 0$)

  $\implies \exists X_\infty$, with $\mathbb{E}(X_\infty) < \infty$ s.t. $X_n \xrightarrow[n\to\infty]{a.s} X_\infty$

- If $(X_n)_{n\geq 0}$ is L^p- bounded for some $p > 1$ $\left(\mathbb{E}(|X_n|^p) \leq c \quad \forall n \geq 0 \right)$

  $\implies \exists X_\infty$ s.t. $X_n \xrightarrow[n\to\infty]{L^p,\, a.s.} X_\infty$

- <u>Probability of fixation</u>

From the definition, it turns that $(X_n)_{n\in\mathbb{N}_0}$ is a <u>bounded martingale</u>. Indeed, $\forall j \in S$ initial state $(X_0 \sim \delta_j)$:

$$\mathbb{E}_j(X_{n+1} | X_n) = \mathbb{E}\left(\text{Bin}\left(N, \frac{X_n}{N}\right) \Big| X_n \right) = X_n \qquad \forall n \in \mathbb{N}_0$$

so that $\mathbb{E}_j(X_n) = \mathbb{E}_j(X_0) = j$

From the martingale convergence theorem:

$\forall p \geq 1$, since (X_n) is bounded

$$X_n \xrightarrow[n\to\infty]{} X_\infty \qquad \mathbb{P}_j\text{- a.s. and in } L^p,$$

with $\mathbb{E}_j(X_\infty) = \mathbb{E}_j(X_0) = j$

$\ast$ If we can argue (future lectures) (as it is intuitive) that $X_\infty \in \{0, N\}$, then $j = \mathbb{E}_j(X_\infty) = N \cdot \mathbb{P}(X_\infty = N)$

$$\implies \mathbb{P}(X_\infty = N) = \frac{j}{N} \quad \text{and} \quad \mathbb{P}(X_\infty = 0) = 1 - \frac{j}{N}$$

Then $j = \mathbb{E}_j(X_\infty) = N\,\mathbb{P}(X_\infty = N)$

$$\Rightarrow \quad \mathbb{P}(X_\infty = N) = \frac{j}{N} \quad \text{and} \quad \mathbb{P}(X_\infty = 0) = 1 - \frac{j}{N}$$

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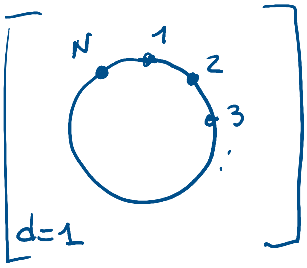
b. Evolution of state (e.g. opinion) among interacting individuals

↳ interacting particle systems

General model assumptions:

- individuals (or particles) interact through a "network of friends" that is described by a graph $G=(V,E)$, where vertices are the position of individuals and edges correspond to friendship among a couple
- the state (or opinion) takes value on finite E so that the whole system is described by states in $S := E^V$ → state or configuration space
- The dynamics change the state of a single individual at each step, and only depends on the state of its neighborhood.

Voter model



- As a graph of interaction, consider $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ and assume that there is a votant at any vertex of $\mathbb{T}_N$ (hence N in total)
- Any votant holds an opinion $\in \{0,1\}$ (favorable / against) so that the whole system of opinions takes value in $S = \{0,1\}^N \ni x = (x(i))_{i\leq N}$ and $x(i) \in \{0,1\}$
- At each time $n \in \mathbb{N}$, an individual, chosen uniformly in $\{1,\dots,N\}$ choose uniformely one of its neighbors and agree with its opinion.

Let X_n = set of opinions after n steps $= (X_n(i))_{i\leq N}$, $\forall n \in \mathbb{N}$

so that $X_n(i)$ = opinion of votant at site i, $i \leq N$

Then $(X_n)_{n\in\mathbb{N}}$ is a M.C. on S.

Then $(X_m)_{m\in\mathbb{N}_0}$ is a MC on S.

Indeed: let $(I_m)_{m\in\mathbb{N}}$ iid $\sim \mathcal{U}(1,\dots,N)$

and $(U_m)_{m\in\mathbb{N}}$ iid $\sim \mathcal{U}(\underbrace{-e_1,+e_1}_{\text{neighborhood in } \mathbb{T}_N})$ with

Then $(X_m)_{m\geq 0}$ is recursively defined so that

$$(*) \qquad X_m(i) = \begin{cases} X_{m-1}(i) & \text{if } I_m \neq i \\ X_{m-1}(\underbrace{I_m+U_m}_{\text{chosen neighbor of } I_m}) & \text{if } I_m = i \end{cases}$$

which is clearly a MC with transition prob. that can be obtained from $(*)$ (try as an exercise).

Remark:

Since the evolution rule "made copy of other opinions", as for the Wright's model there are 2 absorbing states which correspond to the "total agreement", namely the states: $\underline{0} = \{$all votants have opinion $0\}$ and $\underline{1} = \{$all votants have opinion $1\}$

- Probability of total agreement

One can consider the sequence $(M_m)_{m\geq 0}$, where

$$M_m = \sum_{i=1}^{N} X_m(i) = \#\text{ votants with opinion } 1 \in \{1,\dots,N\}$$

Then $(M_m)_{m\geq 0}$ is a sequence of r.v.'s on $\{1,\dots,N\}$

In particular it is a martingale (bounded in L^p, $\forall p\geq 1$) w.r.t. $\mathcal{F}_m = \sigma(X_k, k\leq m)$:

$$\mathbb{E}(M_n \mid \mathcal{F}_{n-1}) = M_{n-1} - \mathbb{E}(X_{n-1}(I_n) \mid \mathcal{F}_{n-1}) + \mathbb{E}(X_{n-1}(I_n + U_n) \mid \mathcal{F}_{n-1})$$

$$= M_{n-1} - \sum_{i=1}^{N} X_{n-1}(i) \cdot \underbrace{\mathbb{P}(I_n = i)}_{= \frac{1}{N}} + \sum_{i=1}^{N} X_{n-1}(i) \underbrace{\mathbb{P}(I_n + U_n = i)}_{= \frac{1}{N}}$$

$$= M_{n-1}$$

By the martingale convergence theorem we then have

$$M_n \xrightarrow[n \to \infty]{} M_\infty \quad \mathbb{P}_j\text{-a.s. and in } L^p, \ \forall j \text{ initial state}, \ \forall p \geq 1$$

so that $\mathbb{E}_j(M_\infty) = \mathbb{E}_j(M_0) = j$

As before, if we can argue that $M_\infty \in \{0, N\}$

$$\Longrightarrow \quad \mathbb{P}_j(M_\infty = N) = \frac{j}{N} \quad \text{and} \quad \mathbb{P}_j(M_\infty = 0) = 1 - \frac{j}{N}$$
