# COMPUTABILITY (12/01/2026)

**EXERCISE 3.2** There exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $|W_{s(x)}| = 2x$ and $|E_{s(x)}| = x$

Idea : function of two arguments

$$g(\underset{\substack{\uparrow \\ \text{fixed}}}{x}, y) = \begin{cases} qt(2,y) & \text{if } y < 2x \\ \uparrow & \text{otherwise} \end{cases}$$

$$= qt(2,y) + \underbrace{\mu z.\ \overbrace{y+1 \dot{-} 2x}^{\substack{0 \text{ if } y<2x \\ >0 \text{ otherwise}}}}_{\substack{0 \quad \text{if } y<2x \\ \uparrow \quad \text{otherwise}}}$$

computable, composition / minimalisation of computable function

By smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable such that
$\forall x, y$ $\qquad \varphi_{s(x)}(y) = g(x,y)$

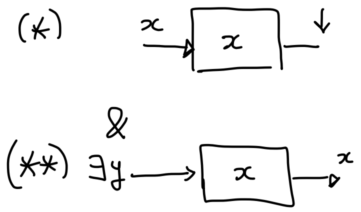
We claim that $s$ is the desired function

$|W_{s(x)}| = |\{ y \mid y < 2x \}| = 2x$

$|E_{s(x)}| = |\{ qt(2,y) \mid y \in W_{s(x)} \}| = |\{ qt(2,x) \mid y < 2x \}|$

$= |\{ z \mid z < x \}| = x$

EXERCISE 8.2 : Classify $A = \{ x \in \mathbb{N} \mid x \in W_x \cap E_x \}$



to check if $x \in A$ I need to verify that $(*)$ $\varphi_x(x) \downarrow$

$(**)$ there is $y$ s.t. $\varphi_x(y) = x$

and both are semidecidable

conjecture :

A r.e. but not recursive

$\hookrightarrow$ $\bar{A}$ not r.e. ($\rightsquigarrow$ not recursive)

A is r.e.

In fact

$$SC_A(x) = \mathbb{1}\left(\mu(y,t).\ H(x,x,t) \wedge S(x,y,x,t)\right)$$
$$= \mathbb{1}\left(\mu\omega.\ H(x,x,(\omega)_2) \wedge S(x,(\omega)_1,x,(\omega)_2)\right)$$

computable

A not recursive $\qquad K \leq_m A$

consider

$$g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$
$$= y \cdot SC_K(x)$$

computable. Hence by smn there is total computable $s : \mathbb{N} \to \mathbb{N}$ such that $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$

s is the reduction function for $K \leq_m A$

* if $x \in K$ then $\varphi_{s(x)}(y) = y \quad \forall y$

Thus $s(x) \in \underbrace{W_{s(x)}}_{\mathbb{N}} \cap \underbrace{E_{s(x)}}_{\mathbb{N}} = \mathbb{N}$, hence $s(x) \in A$.

* if $x \notin K$ then $\varphi_{s(x)}(y) \uparrow \ \forall y$

Thus

$$s(x) \notin \underset{\substack{\parallel \\ \emptyset}}{W_{s(x)}} \cap \underset{\substack{\parallel \\ \emptyset}}{E_{s(x)}} = \emptyset$$

Hence $s(x) \notin A$

Finally $\bar{A}$ not r.e. (otherwise if $\bar{A}$ r.e., since also $A$ is, we would have $A$ recursive, which is not the case).

and therefore $\bar{A}$ not recursive.

$\square$

**EXERCISE 8.10** Classify $A = \{ x \in \mathbb{N} \mid |W_x \cap E_x| = 1 \}$

conjecture:

to check $x \in A$ I need to be sure that $|\{ y \mid y \in W_x \cap E_x \}| = 1$

checking if $y \in W_x \cap E_x$ is semidecidable, but I would need to be sure that the condition holds for exactly one $y$, i.e. for the infinitely many other $y$'s the condition is false. Hence for infinitely many values I have to verify the complement of strictly semidecidable property, hence a property which is not semidecidable.

not decidable

$A, \bar{A}$ not r.e. (hence not recursive)

$A = \{ x \mid \varphi_x \in \mathcal{A} \}$

$\mathcal{A} = \{ f \mid |dom(f) \cap cod(f)| = 1 \}$

A saturated

I try to use Rice - Shapiro

* $A$ not r.e.

consider id identity function $|dom(id) \cap cod(id)| = |\mathbb{N} \cap \mathbb{N}|$

$id \notin A$ $= |\mathbb{N}| \neq 1$

but if $\vartheta(x) = \begin{cases} 0 & \text{if } x = 0 \\ \uparrow & \text{otherwise} \end{cases}$

$\vartheta \subseteq id$ $\quad \vartheta$ finite $\quad \vartheta \in A$

$|dom(\vartheta) \cap cod(\vartheta)| = |\{0\} \cap \{0\}| = |\{0\}| = 1$

hence, by Rice - Shapiro, $A$ not r.e.

• $\bar{A}$ not r.e.

note that $\vartheta$ above $\vartheta \notin \bar{A}$

and $\vartheta' = \emptyset$ $\quad \vartheta' \subseteq \vartheta$ finite $\quad \vartheta' \in \bar{A}$

since $|dom(\vartheta') \cap cod(\vartheta')| = |\emptyset| = 0$

hence, by Rice - Shapiro, $\bar{A}$ not r.e.

<u>EXERCISE</u> : Classify $B = \{x \in \mathbb{N} \mid |W_x \cap E_x| \geq 1\}$

to check if $x \in B$ we just need a single witness $\underbrace{y \in W_x \cap E_x}_{\text{semidecidable}}$

conjecture : $B$ is r.e. not recursive

$\bar{B}$ not r.e. (hence not recursive)

HOME : suggestion

$B$ saturated , $B$ r.e. by showing $sc_B$ computable

$B$ not recursive by Rice.

9.14 Prove that given a total ~~computable~~ function $f: \mathbb{N} \to \mathbb{N}$ injective, the set $C_f = \{x \mid f(x) \in W_x\}$ is not saturated.

<u>idea</u>: find $e \in \mathbb{N}$

$$\varphi_e(y) = \begin{cases} 0 & \text{if } y = f(e) \\ \uparrow & \text{otherwise} \end{cases} \qquad (*)$$

if we show that $(*)$ exists

→ $e \in C_f$ since $f(e) \in W_e$

→ there is $e' \neq e$ s.t. $\varphi_e = \varphi_{e'}$

→ $e' \notin C_f$ $W_{e'} = W_e = \{f(e)\}$ and $f(e) \neq f(e')$ since $f$ is injective

thus $f(e') \notin W_{e'} = \{f(e)\}$

For showing $(*)$, use, as usual smn + 2nd recursion theorem.

EXERCISE: $B = \{x \in \mathbb{N} \mid \exists y \geq x.\ \varphi_x(y) > y\}$

is B saturated?

<u>idea</u>: there is an index $e$ s.t.

$$\varphi_e(y) = e + 1 \qquad (*)$$

if $(*)$ holds then

- $e \in B$ $y = e \geq e$ and $\varphi_e(e) = e+1 > e$
- since there are infinitely many indices for $\varphi_e$ there is $e' \in \mathbb{N}$ $e' > e$ s.t. $\varphi_{e'} = \varphi_e$
- $e' \notin B$ since $\forall y \geq e'$ $\varphi_{e'}(y) = \varphi_e(y) = e+1 \leq e' \leq y$

EXERCISE: RANDOM NUMBERS (from $1^{st}$ lesson)

→ $n \in \mathbb{N}$ is random if all programs producing $n$ in output are "larger" than $n$

two questions:

(1) there are infinitely many random numbers

(2) the property of being random is not decidable

Try again:

→ size of a program? $|P_e| = e$

→ define a number to be random if

for all $e \in \mathbb{N}$ s.t. $\varphi_e(0) = n$ it holds $e > n$

(2) The set $R = \{ n \in \mathbb{N} \mid n \text{ random} \}$ is not recursive

(hence it is necessarily infinite (1))

Assume $R$ is recursive, i.e. $\chi_R(n) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{otherwise} \end{cases}$

Define

$$\begin{aligned} g(n,x) &= \text{least random number} > n \\ &= \mu y.\ y \in R \wedge y > n \\ &= n+1 + \mu z.\ (n+1+z \in R) \\ &= n+1 + \mu z.\ |\chi_R(n+1+z) - 1| \end{aligned}$$

computable.

by smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x, n$

$$\varphi_{s(n)}(x) = g(n,x)$$

By 2nd recursion theorem there is $n_0 \in \mathbb{N}$ s.t. $\varphi_{n_0} = \varphi_{s(n_0)}$

Hence

$$\varphi_{n_0}(x) = \varphi_{s(n_0)}(x) = g(n_0, x) = \text{(least) random number } > n_0$$

i.e. program $n_0$ outputs a number $r > n_0$ which is random

ABSURD.

$\Rightarrow$ $R$ is not recursive.

Note that $\bar{R}$ is r.e.

$$sc_{\bar{R}}(n) = 1\left(\mu t . \bigvee_{e \leq n} S(e, 0, n, t)\right) \quad \text{computable}$$

$$e \leqslant n$$

hence $R$ is not r.e.

THE END