Esercizio 1. Sia W l'insieme dei vettori $(a,b,c,d) \in \mathbb{R}^4$ tali che (a,b) e (c,d) sono ortogonali al vettore $v = (2,-1)$. Sia U l'insieme dei vettori $(a,b,c,d) \in \mathbb{R}^4$ tali che (a,c) e (b,d) appartengono al sottospazio $< v >$.

(a) Mostrare che W è un sottospazio di $\mathbb{R}^4$ e determinare una sua base.

(b) Determinare una base di U.

(c) Determinare una base di $U \cap W$ e di $U + W$.

(d) Dire se esiste un vettore (a,b,c,d) in U o in W tale che $(a,b),(c,d)$ sono linearmente indipendenti (motivare la risposta).

(e) (FACOLTATIVO) Dire se esiste un vettore (a,b,c,d) in $U+W$ tale che $(a,b),(c,d)$ sono linearmente indipendenti.

Svolgimento:

$$W = \left\{ \begin{pmatrix} a\\b\\c\\d \end{pmatrix} \;\middle|\; \begin{pmatrix} a\\b \end{pmatrix} \perp \begin{pmatrix} 2\\-1 \end{pmatrix} \text{ e } \begin{pmatrix} c\\d \end{pmatrix} \perp \begin{pmatrix} 2\\-1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a\\b\\c\\d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} 2a-b=0 \\ 2c-d=0 \end{cases} \right\}$$

$$U = \left\{ \begin{pmatrix} a\\b\\c\\d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} a\\c \end{pmatrix} \in < \begin{pmatrix} 2\\-1 \end{pmatrix} >, \begin{pmatrix} b\\d \end{pmatrix} \in < \begin{pmatrix} 2\\-1 \end{pmatrix} > \right\}$$

a)
$$\begin{pmatrix} a\\b \end{pmatrix} \perp \begin{pmatrix} 2\\-1 \end{pmatrix} \iff \begin{pmatrix} a\\b \end{pmatrix} \cdot \begin{pmatrix} 2\\-1 \end{pmatrix} = 2a-b=0 \qquad b = 2a$$

$$\begin{pmatrix} c\\d \end{pmatrix} \perp \begin{pmatrix} 2\\-1 \end{pmatrix} \iff \begin{pmatrix} c\\d \end{pmatrix} \cdot \begin{pmatrix} 2\\-1 \end{pmatrix} = 2c-d=0 \qquad d = 2c$$

$$\begin{pmatrix} a\\b\\c\\d \end{pmatrix}$$

$$W = \left\{ \begin{pmatrix} a\\2a\\c\\2c \end{pmatrix} \;\middle|\; a,c \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} 1\\2\\0\\0 \end{pmatrix} + c\begin{pmatrix} 0\\0\\1\\2 \end{pmatrix} \;\middle|\; a,c \in \mathbb{R} \right\} =: \boxed{< \begin{pmatrix} 1\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\2 \end{pmatrix} > \leq \mathbb{R}^4}$$

abbiamo dim. inclusione

$$w_i \in \mathbb{R}^4 \quad i = 1, \dots, k$$

$$< w_1, \dots, w_k > \leq \mathbb{R}^4$$

$$\boxed{B_W = \left\{ \begin{pmatrix} 1\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\2 \end{pmatrix} \right\} \quad \dim W = 2}$$

b) Determiniamo una base di U.

$$U = \left\{ \begin{pmatrix} a\\b\\c\\d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} a\\c \end{pmatrix} \in < \begin{pmatrix} 2\\-1 \end{pmatrix} >, \begin{pmatrix} b\\d \end{pmatrix} \in < \begin{pmatrix} 2\\-1 \end{pmatrix} > \right\} = \left\{ \begin{pmatrix} a\\b\\c\\d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \begin{pmatrix} a\\c \end{pmatrix} = \begin{pmatrix} 2s\\-s \end{pmatrix} \text{ con } s \in \mathbb{R} \\ \begin{pmatrix} b\\d \end{pmatrix} = \begin{pmatrix} 2t\\-t \end{pmatrix} \text{ con } t \in \mathbb{R} \end{array} \right\}$$

$$\begin{pmatrix} a\\c \end{pmatrix} \in < \begin{pmatrix} 2\\-1 \end{pmatrix} > \qquad \begin{pmatrix} a\\c \end{pmatrix} = \begin{pmatrix} 2s\\-s \end{pmatrix} \qquad \begin{pmatrix} b\\d \end{pmatrix} \in < \begin{pmatrix} 2\\-1 \end{pmatrix} > \qquad \begin{pmatrix} b\\d \end{pmatrix} = \begin{pmatrix} 2t\\-t \end{pmatrix}$$

$$s\begin{pmatrix} 2\\-1 \end{pmatrix}$$

$$U=\left\{\begin{pmatrix}2s\\2t\\-s\\-t\end{pmatrix} \middle| \; s,t\in\mathbb{R}\right\} = \left\langle \begin{pmatrix}2\\0\\-1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\\-1\end{pmatrix}\right\rangle$$

$$\boxed{B_U=\left\{\begin{pmatrix}2\\0\\-1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\\-1\end{pmatrix}\right\},\ \dim U=2}$$

c) Determiniamo una base $B_{U\cap W}$ di $U\cap W$

$$W=\left\{\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\in\mathbb{R}^4 \middle| \begin{cases}2a-b=0\\2c-d=0\end{cases}\right\} \qquad U=\left\{\begin{pmatrix}2s\\2t\\-s\\-t\end{pmatrix} \middle| \; s,t\in\mathbb{R}\right\}$$

$u\in U \quad u=\begin{pmatrix}2s\\2t\\-s\\-t\end{pmatrix}$ quando $u\in W$? $\Longleftrightarrow \begin{cases}\cancel{2}(2s)-\cancel{2}t=0\\-2s+t=0\end{cases} \quad \begin{cases}t=2s\\t=2s\end{cases}$

$$U\cap W=\left\{\begin{pmatrix}2s\\4s\\-s\\-2s\end{pmatrix} \middle| \; s\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}2\\4\\-1\\-2\end{pmatrix}\right\rangle$$

$$\boxed{B_{U\cap W}=\left\{\begin{pmatrix}2\\4\\-1\\-2\end{pmatrix}\right\},\ \dim(U\cap W)=1}$$

$\Rightarrow$ non sono in somma diretta

Calcoliamo $\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)=2+2-1=3$

$$B_{U+W}=\left\{\begin{pmatrix}1\\2\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\-1\\0\end{pmatrix}\right\},\ \dim(U+W)=3$$

Base di W cerchiamo un vettore $u\in U\setminus W$ $\quad u\in U$

$u\notin\left\langle\begin{pmatrix}2\\4\\-1\\-2\end{pmatrix}\right\rangle=U\cap W$

d) ⊛ Esiste un vettore $u\in U$ tale che $\begin{pmatrix}a\\b\end{pmatrix}$ e $\begin{pmatrix}c\\d\end{pmatrix}$ siano lin. indip.? $\boxed{\text{NO}}$

Esiste un vettore $w\in W$ tale che $\begin{pmatrix}a\\b\end{pmatrix}$ e $\begin{pmatrix}c\\d\end{pmatrix}$ siano lin. indip.? $\boxed{\text{NO}}$

$$U=\left\{\begin{pmatrix}2s\\2t\\-s\\-t\end{pmatrix} \middle| \; s,t\in\mathbb{R}\right\} \qquad \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}2s\\2t\end{pmatrix} \quad \begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}-s\\-t\end{pmatrix}$$

$\det\begin{pmatrix}a&c\\b&d\end{pmatrix}=\det\begin{pmatrix}2s&-s\\2t&-t\end{pmatrix}=$

$=-2st+2st=0$

$\Rightarrow$ sono sempre lin. dipendenti.

$$W=\left\{\begin{pmatrix}a\\2a\\c\\2c\end{pmatrix} \middle| \; a,c\in\mathbb{R}\right\} \qquad \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a\\2a\end{pmatrix} \quad \begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}c\\2c\end{pmatrix}$$

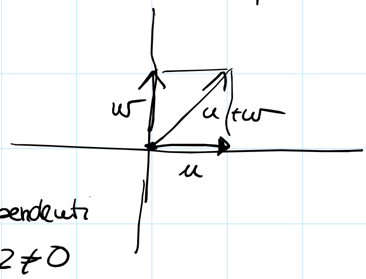
$\det\begin{pmatrix}a&c\\2a&2c\end{pmatrix}=2ac-2ac=0$

$\Rightarrow$ sono sempre lin. dipendenti

e) Esiste $v \in U+W$ tale che $\begin{pmatrix} a \\ b \end{pmatrix}$ e $\begin{pmatrix} c \\ d \end{pmatrix}$ siano lin. indipendenti?

$$\begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ \hline -1 \\ 0 \end{pmatrix} \in U+W$$

$u \in U \setminus W \qquad w \in W \setminus U$



$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ sono lin. indipendenti

$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\det\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} = 2 \neq 0$

Esercizio 2. Sia $f_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ l'endomorfismo la cui matrice associata rispetto alla base canonica di $\mathbb{R}^3$ è la matrice

$$A_a = \begin{pmatrix} 0 & 1 & a-1 \\ 1-a & -1 & 0 \\ 2-2a & 2a & 0 \end{pmatrix}, \quad a \in \mathbb{R}.$$

(a) Stabilire per quali valori di a f_a è un isomorfismo.

(b) Per ogni valore di a calcolare la controimmagine $f_a^{-1}((1,0,0))$.

(c) Stabilire per quali valori di a si ha $\ker f_a \oplus \operatorname{im} f_a = \mathbb{R}^3$.

(d) Posto $a = 1$, stabilire se f_a è diagonalizzabile, e, in caso affermativo, determinare una matrice di cambiamento di base P che diagonalizza f_a, ossia $P^{-1}A_aP$ è diagonale.

Svolgimento: $f_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\quad A_a = \begin{pmatrix} 0 & 1 & a-1 \\ 1-a & -1 & 0 \\ 2-2a & 2a & 0 \end{pmatrix} \quad a \in \mathbb{R}.$

a) f_a è isomorfismo $\Longleftrightarrow \det A_a \neq 0$

$$\det\begin{pmatrix} 0 & 1 & a-1 \\ 1-a & -1 & 0 \\ 2-2a & 2a & 0 \end{pmatrix} = (a-1)\left(\cancel{2a} - 2a^2 + 2 - \cancel{2a}\right) = (a-1)(2-2a^2) =$$

$$= \underline{(a-1)}\,2\,\underline{(1-a^2)}$$

$\det A_a = 0$ se e solo se $\begin{array}{l} a=1 \\ a^2=1 \end{array}$ $\quad \begin{array}{l} a=1 \text{ oppure} \\ a=-1 \end{array}$

$$\boxed{f_a \text{ è isomorfismo} \Leftrightarrow a \in \mathbb{R} \setminus \{-1, 1\}}$$

b) Calcoliamo $f_a^{-1}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ $\forall a \in \mathbb{R}$ $\quad \left(A_a \middle| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right) = \left(\begin{array}{ccc|c} 0 & 1 & a-1 & 1 \\ 1-a & -1 & 0 & 0 \\ 2-2a & 2a & 0 & 0 \end{array}\right)$

$S_{1,2}$ $\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 2-2a & 2a & 0 & 0 \end{array}\right)$ $\xrightarrow{3°R - 2\,1°R}$ $\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & \underline{2a+2} & \underline{0} & \underline{0} \end{array}\right) \xrightarrow{3°R-(2a+2)2°}$

$$\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & -2(a^2-1) & -2a-2 \end{array}\right)$$

$2a+2-(2a+2)\cdot 1=0$

$0-(2a+2)(a-1)=-2(a+1)(a-1)=-2(a^2-1)$

$0-(2a+2)\cdot 1=-2a-2$

Se $a\in\mathbb{R}\setminus\{-1,1\}$

$$\begin{cases}(1-a)\,x-y=0\\ y+(a-1)z=1\\ -2(a^2-1)\,z=-2(a+1)\end{cases}$$

$$\begin{cases}(1-a)x=y=0 & x=0\\ y=-\cancel{(a-1)}\frac{1}{\cancel{a-1}}+1=-1+1=0\\ z=\frac{\cancel{-2}\cancel{(a+1)}}{\cancel{-2}(a-1)\cancel{(a+1)}}=\frac{1}{a-1}\end{cases}$$

$$\boxed{\text{Se } a\in\mathbb{R}\setminus\{-1,1\}\quad f_a^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}=\left\{\begin{pmatrix}0\\0\\\frac{1}{a-1}\end{pmatrix}\right\}}$$

Se $a=-1$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)\qquad \begin{cases}2x-y=0\\ y-2z=1\\ 0=0\end{cases}\qquad \begin{cases}2x=2z+1\\ y=2z+1\end{cases}$$

$$\begin{cases}x=z+\frac{1}{2}\\ y=2z+1\end{cases}$$

$$\boxed{f_{-1}^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}=\left\{\begin{pmatrix}z+1/2\\2z+1\\z\end{pmatrix}\,\middle|\, z\in\mathbb{R}\right\}=\underset{z=0}{\begin{pmatrix}1/2\\1\\0\end{pmatrix}}+\underset{\text{coeff. di } z}{\overset{\text{Ker} f_{-1}}{\left\langle\begin{pmatrix}1\\2\\1\end{pmatrix}\right\rangle}}}$$

Se $a=1$

$$\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & -2(a^2-1) & -2a-2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right)\qquad \begin{cases}-y=0\\ y=1\\ 0=-4\end{cases}\qquad \text{impossibile}\qquad \boxed{f_1^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}=\emptyset}$$

c) Per quali $a\in\mathbb{R}$ $\quad \text{Ker} f_a\oplus \text{Im} f_a=\mathbb{R}^3$?

$\dim\mathbb{R}^3=\dim\text{Ker} f_a+\dim\text{Im} f_a$

$3=\dim\text{Ker} f_a+\dim\text{Im} f_a$

$$\boxed{\begin{array}{l}\text{Se } a\in\mathbb{R}\setminus\{-1,1\}\\ \text{Ker} f_a\oplus\text{Im} f_a=\mathbb{R}^3\\ \{\vec{0}\}\oplus\mathbb{R}^3\end{array}}$$

f_a è isomorfismo $\Longrightarrow$ $\text{Ker} f_a=\{\vec{0}\}$, $\text{Im} f_a=\mathbb{R}^3$ $\qquad \{\vec{0}\}\oplus\mathbb{R}^3=\mathbb{R}^3$

Se $a=-1$ $A_{-1}=\begin{pmatrix}0&1&-2\\2&-1&0\\4&-2&0\end{pmatrix}$ $A_a=\begin{pmatrix}0&1&a-1\\1-a&-1&0\\2-2a&2a&0\end{pmatrix}$

$\text{Ker} f_{-1}$ $\begin{cases}y-2z=0\\2x-y=0\\4x-2y=0\end{cases}$ $\begin{cases}y=2z\\y=2x\end{cases}$ $\begin{cases}y=2z\\x=z\end{cases}$ $\left\{\begin{pmatrix}z\\2z\\z\end{pmatrix}\Big| z\in\mathbb{R}\right\}=\boxed{<\begin{pmatrix}1\\2\\1\end{pmatrix}>=\text{Ker} f_{-1}}$

$\boxed{\text{Im} f_{-1}}=<\begin{pmatrix}0\\2\\4\end{pmatrix},\begin{pmatrix}1\\-1\\-2\end{pmatrix},\begin{pmatrix}-2\\0\\0\end{pmatrix}>=\boxed{<\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}>}$

$\begin{pmatrix}1\\2\\1\end{pmatrix}\notin \text{Im} f_{-1}$

$rg\begin{pmatrix}0&1&1\\1&0&2\\2&0&1\end{pmatrix}=3$ perché

$\det\begin{pmatrix}0&1&1\\1&0&2\\2&0&1\end{pmatrix}=-(1-4)=3\neq 0$

$\boxed{\text{Se } a=-1 \quad \text{Ker} f_{-1}\oplus \text{Im} f_{-1}=\mathbb{R}^3}$

Se $a=1$ $A_1=\begin{pmatrix}0&1&0\\0&-1&0\\0&2&0\end{pmatrix}$ $\text{Ker} f_1$ $\begin{cases}y=0\\-y=0\\2y=0\end{cases}$ $\begin{pmatrix}x\\0\\z\end{pmatrix}$

$\text{Ker} f_1=<\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}>$ $\text{Im} f_1=<\begin{pmatrix}1\\-1\\2\end{pmatrix}>$ $rg\begin{pmatrix}1&0&1\\0&0&-1\\0&1&2\end{pmatrix}=3$

$\Rightarrow \boxed{\text{Ker} f_1\oplus \text{Im} f_1=\mathbb{R}^3}$

Osservazione: $A=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ $\text{Ker} f=<\begin{pmatrix}1\\0\\0\end{pmatrix}>$

$\text{Ker} f$ $\begin{cases}y=0\\z=0\\0=0\end{cases}$ $\begin{pmatrix}x\\\cancel{y}\\\cancel{z}\end{pmatrix}$ $\begin{pmatrix}x\\0\\0\end{pmatrix}$ $\text{Ker} f=<\begin{pmatrix}1\\0\\0\end{pmatrix}>$

$\text{Im} f=<\cancel{\begin{pmatrix}0\\0\\0\end{pmatrix}},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}>=<\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}>$ $\text{Ker} f\subseteq \text{Im} f.$

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d) Diagonalizzare $a=1$ $A_1=\begin{pmatrix}0&1&0\\0&-1&0\\0&2&0\end{pmatrix}$ $rg(A_1)=1$

$\dim \text{Ker} f_1=2 \Rightarrow m_g(0)=2$ $\quad 0, 0, d_3$

$tr(A_1)=0-1+0=-1=d_1+d_2+d_3=0+0+d_3 \Rightarrow d_3=-1$

$P_{A_1}(x)=\det\begin{pmatrix}-x&1&0\\0&-1-x&0\\0&2&-x\end{pmatrix}=-x(-1-x)(-x)=-x^2(x+1)$

$0\ 0\ -1$

A_1 ha due autovalori $m_a(0)=2=m_g(0)$

$m_a(-1) = 1 = m_g(-1)$ perché $1 \leq m_g(d) \leq m_a(d)$

A_1 verifica la prima condizione del Teorema di diagonalizzabilità perché ha tutti autovalori reali.

$m_g(0) := \dim V_0 = \dim \operatorname{Ker}(A_1 - 0\mathbb{I}_3) = \dim \operatorname{Ker} A_1 = 3 - rg(A_1) = 3 - 1 = 2$

$\Rightarrow$ $\boxed{m_a(d) = m_g(d) \quad \forall\, d \text{ autovalore di } A_1 \Rightarrow A_1 \text{ è diagonalizzabile}}$

$A_1 \sim D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

cerchiamo $P = (v_1\, v_2\, v_3) \in GL_3(\mathbb{R})$ tale che $P^{-1} A_1 P = D$

$\boxed{\langle v_1, v_2 \rangle = V_0 = \operatorname{Ker} A}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ $\boxed{V_0 = \langle \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \rangle}$

$\boxed{\langle v_3 \rangle = V_{-1} = \operatorname{Ker}(A + \mathbb{I}_3)}$ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ $\begin{cases} x + y = 0 \\ 0 = 0 \\ 2y + z = 0 \end{cases}$ $\begin{cases} x = -y \\ z = -2y \end{cases}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\begin{pmatrix} -y \\ y \\ -2y \end{pmatrix}$

$\boxed{V_{-1} = \langle \underset{v_3}{\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}} \rangle}$ $\boxed{P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}}$

$P = T^{\mathcal{E}}_{B}$ $\quad B = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\}$

Esercizio 3. Nello spazio euclideo tridimensionale, si considerino la retta

$$r = (2, 0, 3) + < (4, 5, 1) >$$

ed i piani

$$\pi_1 : (0, 0, 1) + < (1, 1, 1), (1, 2, -2) > \quad \text{e} \quad \pi_2 : x - y + z = 2.$$

(a) Determinare una forma parametrica della retta t ortogonale al piano π_1 e passante per il punto $P = (1, 2, 2)$.

(b) Determinare una forma parametrica della retta $s = \pi_1 \cap \pi_2$.

(c) Determinare posizione reciproca e distanza tra le rette r e s.

(d) Determinare tutti i punti Q dello spazio euclideo, per i quali

$$\text{dist}(Q, r) = \text{dist}(Q, s)$$

NOTA: Si consiglia di disegnare il problema.

Svolgimento: $t \perp \pi_1 \quad P \in t \quad P = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \pi_1 : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \rangle$

1) $t: P + \langle v \rangle = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \langle v \rangle$

$v \in \langle \underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}} \rangle^{\perp}$ $\quad v \neq \vec{0}$

$v_1 \times v_2 \in \langle v_1, v_2 \rangle^{\perp}$

$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\begin{cases} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0 \end{cases}$ $\quad a + b + c = 0$, $\quad a + 2b - 2c = 0$

risolvere

$-4 + 3 + 1 = 0$

$-4 + 6 - 2 = 0$

$$v=\begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\-2\end{pmatrix}=\det\begin{pmatrix}1&1&e_1\\1&2&e_2\\1&-2&e_3\end{pmatrix}=\begin{pmatrix}-4\\3\\1\end{pmatrix}\qquad \boxed{t:\begin{pmatrix}1\\2\\2\end{pmatrix}+\left\langle\begin{pmatrix}-4\\3\\1\end{pmatrix}\right\rangle}$$

b) $s=\pi_1\cap\pi_2$ in forma parametrica

$$\pi_1:\begin{pmatrix}0\\0\\1\end{pmatrix}+\left\langle\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\-2\end{pmatrix}\right\rangle\qquad \pi_2: x-y+z=2$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix}+a\begin{pmatrix}1\\1\\1\end{pmatrix}+b\begin{pmatrix}1\\2\\-2\end{pmatrix}=\begin{pmatrix}a+b\\a+2b\\1+a-2b\end{pmatrix}\qquad \cancel{a}+b-\cancel{a}-2b+1+a-2b=2$$

$$a-3b=1\qquad a=3b+1$$

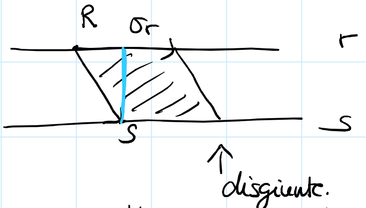
$$\pi_1\cap\pi_2=\left\{\begin{pmatrix}3b+1+b\\3b+1+2b\\1+3b+1-2b\end{pmatrix}\middle|\, b\in\mathbb{R}\right\}=\left\{\begin{pmatrix}4b+1\\5b+1\\b+2\end{pmatrix}\middle|\, b\in\mathbb{R}\right\}=\boxed{\underbrace{\begin{pmatrix}1\\1\\2\end{pmatrix}}_{b=0}+\left\langle\begin{pmatrix}4\\5\\1\end{pmatrix}\right\rangle=s}$$

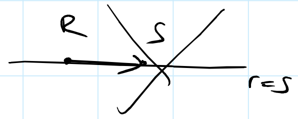
c) Determinare posizione reciproca e distanza fra r ed s.

$$r:\underbrace{\begin{pmatrix}2\\0\\3\end{pmatrix}}_{R}+\left\langle\begin{pmatrix}4\\5\\1\end{pmatrix}\right\rangle\qquad s:\underbrace{\begin{pmatrix}1\\1\\2\end{pmatrix}}_{S}+\left\langle\begin{pmatrix}4\\5\\1\end{pmatrix}\right\rangle$$

$$V_r=\left\langle\begin{pmatrix}4\\5\\1\end{pmatrix}\right\rangle=V_s\Longrightarrow\boxed{r/\!/s}\qquad \dim(r\vee s)=\dim\left(\underbrace{\begin{pmatrix}2\\0\\3\end{pmatrix}}_{R}+\left\langle\underbrace{\begin{pmatrix}4\\5\\1\end{pmatrix}}_{v_r},\underbrace{\begin{pmatrix}4\\5\\1\end{pmatrix}}_{v_s},\underbrace{\begin{pmatrix}-1\\1\\-1\end{pmatrix}}_{S-R}\right\rangle\right)=2$$

$$\Longrightarrow\boxed{r\cap s=\emptyset}$$



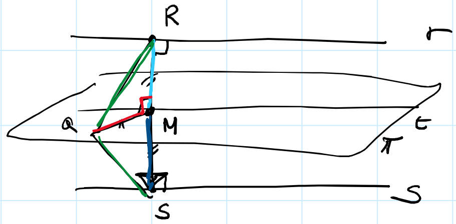


$$d(r,s)=\frac{\|(S-R)\times v_r\|}{\|v_r\|}=\frac{\left\|\begin{pmatrix}-1\\1\\-1\end{pmatrix}\times\begin{pmatrix}4\\5\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}4\\5\\1\end{pmatrix}\right\|}=\frac{\left\|\det\begin{pmatrix}-1&4&e_1\\1&5&e_2\\-1&1&e_3\end{pmatrix}\right\|}{\sqrt{16+25+1}}=\frac{\left\|\begin{pmatrix}6\\-3\\-9\end{pmatrix}\right\|}{\sqrt{42}}=$$

$$=\frac{\left\|3\begin{pmatrix}2\\-1\\-3\end{pmatrix}\right\|}{\sqrt{42}}=\frac{3\sqrt{4+1+9}}{\sqrt{42}}=3\sqrt{\frac{14}{42}}=3\sqrt{\frac{1}{3}}=\boxed{\sqrt{3}=d(r,s)}$$

r è parallela ad s e disgiunta da s.

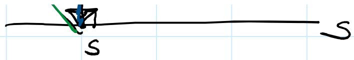
d) Q $\qquad d(Q,r)=d(Q,s)$



1° metodo

Q ∈ piano π passante per $M=\underline{R+S}$ con (R, S) CMD tra r ed s

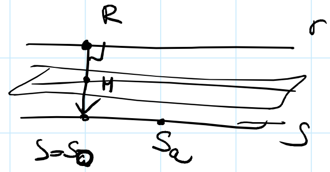
I metodo



$Q \in$ piano π passante per $M = \frac{R+S}{2}$ con (R, S) CMD tra r ed s

$\pi \perp S-R$

$r: \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \right\rangle$ $\quad s: \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \right\rangle$



$R = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ $\quad S_a = \begin{pmatrix} 1+4a \\ 1+5a \\ 2+a \end{pmatrix}$

$S_a - R = \begin{pmatrix} 4a-1 \\ 5a+1 \\ a-1 \end{pmatrix}$ $\quad (S_a - R) \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0$ $\quad 4(4a-1)+5(5a+1)+a-1=0$

$16a - \cancel{4} + 25a + \cancel{5} + a - \cancel{1} = 0 \quad 42a = 0$

$a = 0$

(R, S_0) $\quad R = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ $\quad S = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\quad \boxed{S-R = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}$ $\quad \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = -4+5-1=0$

$\pi: -x+y-z=k$

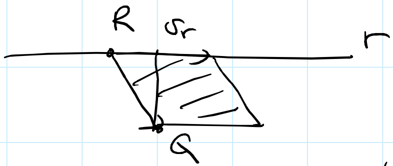
$-\frac{3}{2}+\frac{1}{2}-\frac{5}{2}=-\frac{7}{2}$

$M = \frac{R+S}{2} = \begin{pmatrix} 3/2 \\ 1/2 \\ 5/2 \end{pmatrix}$

$\pi: -x+y-z=-\frac{7}{2} \quad \cdot(-2)$

$\boxed{\pi: 2x-2y+2z=7}$

$Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad d(Q,r) = d(Q,s)$



$d(Q,r) = \frac{\|(Q-R) \times v_r\|}{\|v_r\|}$ $\quad v_r = v_s = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

$d(Q,s) = \frac{\|(Q-S) \times v_r\|}{\|v_r\|}$

Esercizio 2. Data l'applicazione lineare $f_s : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ definita da:

$$f_s(x, y, z) = (4x + (s-4)y, 2x + (s-2)y, (s-2)x + (2-s)y + sz),$$

i) scrivere la matrice A_s associata ad f_s rispetto alla base canonica di $\mathbb{R}^3$;

ii) determinare nucleo ed immagine di f_s al variare del parametro reale s;

iii) determinare, al variare di $s \in \mathbb{R}$, un sottospazio T_s di $\mathbb{R}^3$ tale che $\ker f_s \oplus T_s = \mathbb{R}^3$;

iv) stabilire per quali valori del parametro reale s la matrice A_s è diagonalizzabile su $\mathbb{R}$;

v) per ognuno dei valori trovati in *i)* determinare una base $\mathcal{B}_s$ di $\mathbb{R}^3$ costituita da autovettori di A_s;

vi) posto $s = 4$ si ortonormalizzi la base $\mathcal{B}_4$. Si ottiene in questo modo una base ortonormale di $\mathbb{R}^3$ costituita da autovettori di A_4?

vii) Determinare la matrice associata ad f_s rispetto alla base canonica nel dominio e alla base $\mathcal{B} = \{(1,1,1), (1,1,0), (1,0,0)\}$ nel codominio.

Svolgimento:

$$f_s\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x + (s-4)y \\ 2x + (s-2)y \\ (s-2)x + (2-s)y + sz \end{pmatrix} \qquad f_s : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

i) $$\boxed{A_s = A_{\mathcal{E},\mathcal{E},f_s} = \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ s-2 & 2-s & s \end{pmatrix}}$$

ii) Determinare $\ker f_s$ e $\operatorname{Im} f_s$ al variare di $s \in \mathbb{R}$.

$\det A_s = s(4s - \cancel{8} - 2s + \cancel{8}) = 2s^2$

$\det A_s \neq 0 \iff s \in \mathbb{R} \setminus \{0\}$

Quindi $\forall\, s \in \mathbb{R}\setminus\{0\}$ $\quad \ker f_s = \{\vec{0}\}$ $\quad \operatorname{Im} f_s = \mathbb{R}^3$ perché f_s è isomorfismo.

$\dim \ker f_s = 0 \qquad \dim \operatorname{Im} f_s = 3$

$B_{\ker f_s} = \emptyset \qquad B_{\operatorname{Im} f_s} = \{e_1, e_2, e_3\}$

Per $s = 0$ $\quad A_0 = \begin{pmatrix} 4 & -4 & 0 \\ 2 & -2 & 0 \\ -2 & 2 & 0 \end{pmatrix}$

$\operatorname{rg} A_0 = 1 \quad \operatorname{Im} f_0 = \left\langle \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}, \cancel{\begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}}, \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \right\rangle = \left\langle \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$\dim \operatorname{Im} f_0 = 1$

$\ker f_0 \quad \begin{cases} 4x - 4y = 0 \\ 2x - 2y = 0 \\ -2x + 2y = 0 \end{cases} \qquad x = y \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \left\{ \begin{pmatrix} y \\ y \\ z \end{pmatrix} \,\middle|\, y, z \in \mathbb{R} \right\}$

$$\boxed{\ker f_0 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \dim \ker f_0 = 2}$$

$3 = \dim \mathbb{R}^3 = \dim \ker f_0 + \dim \operatorname{Im} f_0 = 2 + 1$

3) Determinare $T_s \leq \mathbb{R}^3$: $\ker f_s \oplus T_s = \mathbb{R}^3$ al variare $s \in \mathbb{R}$.

3) Determinare $T_s \leq \mathbb{R}^3$: $\operatorname{Ker} f_s \oplus T_s = \mathbb{R}^3$ al variare $s \in \mathbb{R}$.

Se $s \in \mathbb{R} \setminus \{0\}$ $\operatorname{Ker} f_s = \{\vec{0}\}$ $\Rightarrow T_s = \mathbb{R}^3$

Se $s = 0$ $\operatorname{Ker} f_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ $T_0 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$$\operatorname{rg}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \cancel{0} \end{pmatrix} = 3 \qquad T_0' = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

4) $A_s = \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ s-2 & 2-s & s \end{pmatrix}$

$$p_{A_s}(x) = \det(A_s - x I_3) = \det\begin{pmatrix} 4-x & s-4 & 0 \\ 2 & s-2-x & 0 \\ s-2 & 2-s & s-x \end{pmatrix} =$$

$$= (s-x)\left[(4-x)(s-2-x) - 2(s-4)\right] =$$

$$= (s-x)\left[4s - \cancel{8} - 4x - sx + 2x + x^2 - 2s + \cancel{8}\right] =$$

$$= (s-x)\left[x^2 - (s+2)x + 2s\right]$$

$p_{A_s}(x) = 0 \iff$ $x = s$ oppure $x^2 - (s+2)x + 2s = 0$

$$x_{1,2} = \frac{s+2 \pm \sqrt{s^2 + 4s + 4 - 8s}}{2} = \frac{s+2 \pm \sqrt{s^2 - 4s + 4}}{2} =$$

$$= \frac{s+2 \pm \sqrt{(s-2)^2}}{2} = \begin{cases} \frac{s+\cancel{2}+s-\cancel{2}}{2} = \frac{2s}{2} = s \\ \frac{\cancel{s}+2-\cancel{s}+2}{2} = \frac{4}{2} = 2 \end{cases}$$

Gli zeri di $p_{A_s}(x)$ sono

$s, s, 2$ $\quad s=2 \quad 2 \quad 2 \; 2$

$\Rightarrow$ $\boxed{\text{La matrice } A_s \text{ ha tutti autovalori reali}}$

1° caso $\forall s \in \mathbb{R} \setminus \{2\}$ la matrice A_s ha autovalori $s; 2$

$m_a(s) = 2$

$m_a(2) = 1 = m_g(2)$ perché $1 \leq m_g(d) \leq m_a(d)$

Calcoliamo $m_g(s) := \dim V_s = \dim \operatorname{Ker}(A_s - s I_3) = 3 - \operatorname{rg}(A_s - s I_3) = 3 - 1 = 2$

$$A_s = \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ s-2 & 2-s & s \end{pmatrix} \qquad A_s - s I_3 = \begin{pmatrix} 4-s & s-4 & 0 \\ 2 & -2 & 0 \\ s-2 & 2-s & 0 \end{pmatrix}$$

$$\boxed{\begin{matrix} m_a(s)=2=m_g(s) \\ m_a(2)=1=m_g(2) \end{matrix} \Longrightarrow A_s \text{ è diagonalizzabile } \forall s \in \mathbb{R}\setminus\{2\}}$$

2° caso: $s=2$ $\quad A_2 = \begin{pmatrix} 4 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ha un unico autovalore

$m_a(2)=3 \qquad m_g(2) = \dim V_2 = \dim \operatorname{Ker}(A_2 - 2I_3) = 3 - rg\begin{pmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 3-1=2$

$$\boxed{m_g(2)=2 \neq m_a(2)=3} \Longrightarrow \boxed{A_2 \text{ NON è diagonalizzabile.}}$$

5) $\forall s$: A_s sia diagonalizzabile determinare B_s base di autovettori.

$\forall s \in \mathbb{R}\setminus\{2\} \qquad m_a(s)=2=m_g(s) \qquad A_s \sim D_s = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$m_a(2)=1=m_g(2)$

$B_s = \{v_1, v_2, v_3\} \qquad V_s = \langle v_1, v_2 \rangle \qquad V_2 = \langle v_3 \rangle \qquad P_s \in GL_3(\mathbb{R})$:

$P_s^{-1} A_s P_s = D_s \qquad P_s = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$

$\qquad\qquad\qquad\qquad\qquad\qquad s \quad s \quad 2$

Calcoliamo V_s

$V_s = \operatorname{Ker}(A_s - sI_3) = \operatorname{Ker}\begin{pmatrix} 4-s & s-4 & 0 \\ 2 & -2 & 0 \\ s-2 & 2-s & 0 \end{pmatrix} \quad \begin{cases} (4-s)x + (s-4)y = 0 \\ 2x - 2y = 0 \\ (s-2)x + (2-s)y = 0 \end{cases} \quad x=y \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$V_s = \left\{ \begin{pmatrix} y \\ y \\ z \end{pmatrix} \middle| y, z \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Calcoliamo V_2

$V_2 = \operatorname{Ker}(A_s - 2I_3) = \operatorname{Ker}\begin{pmatrix} 2 & s-4 & 0 \\ 2 & s-4 & 0 \\ s-2 & 2-s & s-2 \end{pmatrix} \qquad \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ s-2 & 2-s & s \end{pmatrix} = A_s$

$\begin{cases} 2x + (s-4)y = 0 \\ \cancel{2x + (s-4)y = 0} \\ (s-2)x - (s-2)y + (s-2)z = 0 \end{cases} \qquad \begin{cases} 2x = -(s-4)y \\ x - y + z = 0 \end{cases} \qquad \begin{cases} x = \frac{4-s}{2}y \\ z = -x+y = -\frac{4-s}{2}y + y = \end{cases}$

$= y\left(\frac{-4+s+2}{2}\right) = y\left(\frac{s-2}{2}\right)$

$s \neq 2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad V_2 = \left\{ \begin{pmatrix} \frac{4-s}{2}y \\ y \\ \frac{s-2}{2}y \end{pmatrix} \middle| y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} \frac{4-s}{2} \\ 1 \\ \frac{s-2}{2} \end{pmatrix} \right\rangle = \boxed{\left\langle \begin{pmatrix} 4-s \\ 2 \\ s-2 \end{pmatrix} \right\rangle} \qquad v_3 = \begin{pmatrix} 4-s \\ 2 \\ s-2 \end{pmatrix}$

$$\boxed{B_s = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4-s \\ 2 \\ s-2 \end{pmatrix} \right\} \text{ per } s \neq 2}$$

$\det\begin{pmatrix} 1 & 0 & 4-s \\ 1 & 0 & 2 \\ 0 & 1 & s-2 \end{pmatrix} = -(2-4+s) = -(s-2) \neq 0$

$B_s = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\2\\s-2\end{pmatrix} \right\}$ per $s \neq 2$ $\det\begin{pmatrix}1 & 0 & 0\\1 & 0 & 2\\0 & 1 & s-2\end{pmatrix} = -(2-4+s) = -(s-2) \neq 0$

---

6) Posto $s=4$ ortonormalizzare B_4

$$B_4 = \left\{ \underset{v_1}{\begin{pmatrix}1\\1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\0\\1\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}0\\2\\2\end{pmatrix}} \right\}$$ applichiamo G-S. $B = \{u_1, u_2, u_3\}$

$$\boxed{u_1 = \frac{v_1}{\|v_1\|}} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$$

$$\boxed{w_2 = v_2 - (v_2 \cdot u_1)\, u_1} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \Big[\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}}_{0+0+0=0}\Big] \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$ $$\boxed{u_2 = \frac{w_2}{\|w_2\|}} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\boxed{w_3 = v_3 - (v_3 \cdot u_1)\,u_1 - (v_3 \cdot u_2)\,u_2} = \begin{pmatrix}0\\2\\2\end{pmatrix} - \Big[\underbrace{\begin{pmatrix}0\\2\\2\end{pmatrix} \cdot \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}}_{\sqrt{2}}\Big] \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} - \Big[\underbrace{\begin{pmatrix}0\\2\\2\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}}_{2}\Big]\begin{pmatrix}0\\0\\1\end{pmatrix} =$$
$$= \begin{pmatrix}0\\2\\2\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\boxed{u_3 = \frac{w_3}{\|w_3\|}} = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$$ $$\boxed{B = \left\{ \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \underset{u_3}{\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}} \right\}}$$

$V_4 = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$

$V_2 = \left\langle \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle$

$u_3 \notin V_4$ $u_3 \notin V_2$ $\Rightarrow$ B NON è base di autovettori.

$$A_4 = \begin{pmatrix}4 & 0 & 0\\2 & 2 & 0\\2 & -2 & 4\end{pmatrix}$$ $$\begin{pmatrix}4 & s-4 & 0\\2 & s-2 & 0\\s-2 & 2-s & s\end{pmatrix} = A_s$$

Notiamo $A_4 \neq A_4^t \Rightarrow B$ non è una base di autovettori.

---

6)

$M_s = A_{\mathcal{E}, B, f_s}$ $B = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$ $A_s = A_{\mathcal{E},\mathcal{E},f_s}$

$$\begin{tikzcd}
(\mathbb{R}^3, \mathcal{E}) \arrow[r, "M_s"] \arrow[d, "="'] & (\mathbb{R}^3, B) \\
(\mathbb{R}^3, \mathcal{E}) \arrow[r, "A_s"] & (\mathbb{R}^3, \mathcal{E}) \arrow[u, "T_{\mathcal{E}}^{B}"']
\end{tikzcd}$$

$T_{\mathcal{E}}^{\mathcal{E}} = \mathbb{I}_3$

$M_s = T_{\mathcal{E}}^{B} A_s \mathbb{I}_3 = T_{\mathcal{E}}^{B} A_s$

$$\boxed{M_s = T_{\mathcal{E}}^{B} A_s} = \begin{pmatrix}0 & 0 & 1\\0 & 1 & -1\\1 & -1 & 0\end{pmatrix} \begin{pmatrix}4 & s-4 & 0\\2 & s-2 & 0\\s-2 & 2-s & s\end{pmatrix}$$

$T_{\mathcal{E}}^{B} = \left(T_{B}^{\mathcal{E}}\right)^{-1}$

$T_{B}^{\mathcal{E}} = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{pmatrix}$ $\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\end{array}\right)$

$$T_B^{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$(T_B^{\mathcal{E}} \mid I_3)$ riduciamo $(I_3 \mid T_{\mathcal{E}}^B$.

$$\begin{array}{c} 3^\circ \\ 2^\circ - 3^\circ \\ 1^\circ - 2^\circ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \qquad T_{\mathcal{E}}^B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Esercizio 4. Determinare tutti i sottospazi U di $\mathbb{R}^3$ tali che la proiezione ortogonale di $(1,2,0)$ su U sia $(1,1,1)$.

Svolgimento: detta P_U la proiezione ortogonale su U

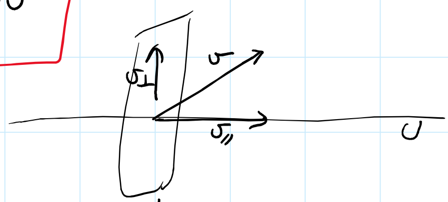
$$P_U\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad U \oplus U^\perp = \mathbb{R}^3$$

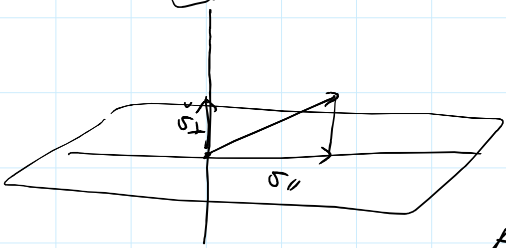
$v = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\quad v = v_\parallel + v_\perp$ $\quad \begin{cases} v_\parallel \in U \\ v_\perp \in U^\perp \end{cases}$ $\quad P_U(v) = v_\parallel$

$v_\parallel = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad v_\perp = v - v_\parallel = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$

$$\boxed{\begin{cases} \begin{pmatrix}1\\1\\1\end{pmatrix} \in U \\ \begin{pmatrix}0\\1\\-1\end{pmatrix} \in U^\perp \end{cases}}$$

notiamo che $\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} = 0+1-1 = 0$





$\begin{cases} \begin{pmatrix}1\\1\\1\end{pmatrix} \in U \\ \begin{pmatrix}0\\1\\-1\end{pmatrix} \in U^\perp \end{cases}$ $\quad U \oplus U^\perp = \mathbb{R}^3$ $\quad \begin{pmatrix}1\\1\\1\end{pmatrix} \in U \Rightarrow \dim U \geq 1$, $\begin{pmatrix}0\\1\\-1\end{pmatrix} \in U^\perp \Rightarrow \dim U^\perp \geq 1$

$\Rightarrow$ A $\boxed{\begin{array}{l}\dim U = 1 \\ \dim U^\perp = 2\end{array}}$ oppure B $\boxed{\begin{array}{l}\dim U = 2 \\ \dim U^\perp = 1\end{array}}$

(A) $\boxed{\text{Se } \dim U = 1 \quad U = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle}$ perché $\begin{pmatrix}1\\1\\1\end{pmatrix} \in U$ e $\dim U = 1$

(B) Se $\dim U = 2$ allora $\dim U^\perp = 1$ $\quad \begin{pmatrix}0\\1\\-1\end{pmatrix} \in U^\perp \Rightarrow U^\perp = \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle$

$U = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle$ $\quad U: y - z = 0 \ \begin{pmatrix}x\\y\\z\end{pmatrix}$

Esercizio 3. Nello spazio euclideo di dimensione tre si considerino i punti $A = (1,1,1)$, $B = (2,1,0)$, $C = (2,1,2)$.

a) Mostrare che i punti A, B, C non sono allineati e determinare il piano π che li contiene;

Esercizio 3. Nello spazio euclideo di dimensione tre si considerino i punti $A = (1,1,1)$, $B = (2,1,0)$, $C = (2,1,2)$.

a) Mostrare che i punti A, B, C non sono allineati e determinare il piano π che li contiene;

b) determinare, se possibile, un punto D tale che il quadrilatero di vertici A, B, C, D sia un quadrato;

c) determinare la retta r ortogonale al piano π e passante per B;

d) calcolare la distanza della retta r dalla retta s passante per $(0,0,0)$ e parallela al vettore $v = (1,2,-1)$;

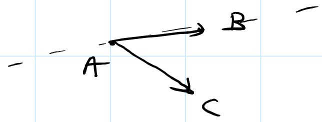
e) determinare il luogo dei punti equidistanti dai punti A, B, C.

Svolgimento:

a) $A=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $B=\begin{pmatrix}2\\1\\0\end{pmatrix}$ $C=\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\boxed{A,B,C \text{ non sono allineati } \iff \dim(A\vee B\vee C)=2}$

$\pi = A\vee B\vee C = A + \langle B-A,\ C-A\rangle$



$\boxed{\pi:\ \begin{pmatrix}1\\1\\1\end{pmatrix}+\left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\rangle \qquad \dim\pi=2}$

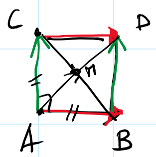
$\Rightarrow$ non sono allineati

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+a+b\\1\\1-a+b\end{pmatrix}$ $\begin{cases}x=1+a+b\\y=1\\z=1-a+b\end{cases}$ $\boxed{\pi:\ \boxed{y=1}}$

b) $A=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $B=\begin{pmatrix}2\\1\\0\end{pmatrix}$ $C=\begin{pmatrix}2\\1\\2\end{pmatrix}$ $\boxed{B-A}=\begin{pmatrix}1\\0\\-1\end{pmatrix}=v_1$

$\boxed{C-A}=\begin{pmatrix}1\\0\\1\end{pmatrix}=v_2$

$\boxed{v_1\cdot v_2=0 \quad v_1\perp v_2 \quad \|v_1\|=\|v_2\|=\sqrt{2}}$



$D=B+(C-A)=\begin{pmatrix}2\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}$

$\boxed{C-A=D-B}$

$\boxed{D=C+(B-A)=\begin{pmatrix}2\\1\\2\end{pmatrix}+\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}}$

$\boxed{A=\begin{pmatrix}1\\1\\1\end{pmatrix}\ B=\begin{pmatrix}2\\1\\0\end{pmatrix}\ C=\begin{pmatrix}2\\1\\2\end{pmatrix}\ D=\begin{pmatrix}3\\1\\1\end{pmatrix}}$

$\Rightarrow$ ABDC è un quadrato perché $\boxed{B-A=D-C}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\|B-A\|=\|C-A\|$ $(B-A)\perp(C-A)$

$D=A+2(M-A)$ $M=\frac{B+C}{2}$

c) determinare $r\perp\pi$ e passante per $B=\begin{pmatrix}2\\1\\0\end{pmatrix}$ $\pi:\ y=1$

$\boxed{r:\ \begin{pmatrix}2\\1\\0\end{pmatrix}+\left\langle\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle}$ $\begin{cases}x=2\\z=0\end{cases}$

$ax+by+cz=d$

$V_\pi^\perp=\left\langle\begin{pmatrix}a\\b\\c\end{pmatrix}\right\rangle$

S R

$s:\ \begin{pmatrix}0\\0\\0\end{pmatrix}+\left\langle\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\rangle$ $r:\ \begin{pmatrix}2\\1\\0\end{pmatrix}+\left\langle\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle$

$S-R$ v_r v_s

$s: \begin{pmatrix}0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle \qquad r: \begin{pmatrix}2\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

r e s sono sghembe se $\dim(r \vee s) = 3$

r e s sono incidenti se $\dim(r \vee s) = 2$

$\Rightarrow$ r ed s sono sghembe.

$S-R \quad v_r \quad v_s$

$r \vee s: \begin{pmatrix}0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}-2\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle$

$\det\begin{pmatrix}-2 & 0 & 1\\ -1 & 1 & 2\\ 0 & 0 & -1\end{pmatrix} = -1(-2) = 2 \neq 0$

$$d(r,s) = \frac{|\det(S-R \ \ v_r \ \ v_s)|}{\|v_r \times v_s\|} = \frac{\left|\det\begin{pmatrix}-2 & 0 & 1\\ -1 & 1 & 2\\ 0 & 0 & -1\end{pmatrix}\right|}{\left\|\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\2\\-1\end{pmatrix}\right\|} = \frac{2}{\left\|\det\begin{pmatrix}0 & 1 & e_1\\ 1 & 2 & e_2\\ 0 & -1 & e_3\end{pmatrix}\right\|} =$$

$$= \frac{2}{\left\|\begin{pmatrix}-1\\0\\-1\end{pmatrix}\right\|} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

$\boxed{d(r,s) = \sqrt{2}}$

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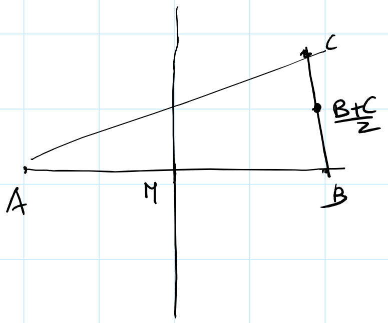
e) Determinare il luogo dei punti equidistanti da A, B, C.



luogo dei punti Q: $d(Q,A) = d(Q,B) =$ asse del segmento A,B $\pi_{A,B}: \frac{A+B}{2} + \langle B-A \rangle^{\perp}$

$Q: \boxed{d(Q,A) = d(Q,B)} \iff Q \in \pi_{A,B}: \frac{A+B}{2} + \langle B-A \rangle^{\perp}$

$Q \quad \boxed{d(Q,B) = d(Q,C)} \iff Q \in \pi_{B,C}: \frac{B+C}{2} + \langle C-B \rangle^{\perp}$

$\frac{A+B}{2} + \langle B-A \rangle^{\perp} \qquad A = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad B = \begin{pmatrix}2\\1\\0\end{pmatrix} \quad C = \begin{pmatrix}2\\1\\2\end{pmatrix}$

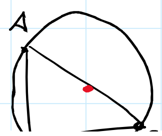
$\begin{pmatrix}3/2\\1\\1/2\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\rangle^{\perp} \qquad x - z = k, \quad \frac{3}{2} - \frac{1}{2} = k \qquad \pi_{A,B}: x - z = 1$

$\frac{B+C}{2} + \langle C-B \rangle^{\perp}$

$\begin{pmatrix}2\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\2\end{pmatrix} \right\rangle^{\perp} \qquad z = k, \quad 1 = k \qquad \pi_{B,C}: z = 1$

$\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle^{\perp}$

$\begin{cases} x - z = 1 \end{cases}$



$$\boxed{\begin{cases} x - z = 1 \\ z = 1 \end{cases}}$$