2° LECTURE - ADVANCED STOCHASTIC PROCESSES

Recall: $(X_n)_{n\geq 0}$ MC with initial measure $\mu \in \mathcal{P}(S)$ and transition matrix $P = (P_{x,y})_{x,y\in S}$ (in short: MC (μ, P)) has law $\mathbb{P}_\mu$ identified by the joint law $\underbrace{(X_{n_1}, \dots, X_{n_k})}_{\text{finite dimension distribution}}$ $\forall k \geq 0$, $\forall n_1, \dots, n_k \geq 0$

Notation

- The average with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$
  (to avoid confusion, if $Y \sim \mu$, then is average is denoted $\mu(Y)$)
- If $\mu = \delta_x$, with $x \in S$, we write $\mathbb{P}_x$ instead of $\mathbb{P}_{\delta_x}$
  ↳ Example: consider the random walk $(S_n)_{n\geq 0}$ s.t.
  $S_0 = x$, $S_n = x + \sum_{k=1}^{n} \xi_k$, with $x \in S$

From the Markov property, it is easy to prove:

$$\mathbb{P}_\mu(X_{n+m} = y \mid X_m = x) \underset{\text{homogeneity}}{=} \mathbb{P}_\mu(X_n = y \mid X_0 = x) = \underline{\mathbb{P}_x(X_n = y)} = \sum_{x_1 \in S} \cdots \sum_{x_{n-1}\in S} P_{x x_1} \cdots P_{x_{n-1} y}$$

decomposing along all possible trajectories $(x,0) \to (y,n)$

(B) $\quad = (P^n)_{x,y} =: \underline{P^{(n)}_{x,y}} \implies P^{(n)}_{x,\cdot}$ is the distribution of $X_n \mid X_0 = x$

In particular, $\forall n \in \mathbb{N}$, P^n is called $\underline{\text{n-steps transition matrix}}$.

If we think at P^n as a linear operator acting on functions $f: S \to \mathbb{R}$, we are led to the following definition.

Def Given a MC $(X_n)_{n\geq 0}$ with transition matrix P, the associated (Markovian) semigroup is the family of operators $(P^n)_{n\in\mathbb{N}}$, on the set of real functions on S, i.e. $P^n: \mathbb{R}^S \to \mathbb{R}^S$, $f \mapsto P^n f$,

with $$\boxed{P^n f(x) = \sum_{y\in S} P^{(n)}_{x,y} f(y)}$$.

In particular, from (B)

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- $$P^n f(x) = \sum_{y\in S} \mathbb{P}_x(X_n=y)\, f(y) = \mathbb{E}_x\big(f(X_n)\big)$$
- $(P^n)_{n\geq 0}$ is indeed a semigroup:

$$P^0 = Id \qquad \text{and} \qquad P^{n+m} = P^n \cdot P^m$$

Check: $$P^{(n+m)}_{x,y} = \mathbb{P}_x(X_{n+m}=y) = \sum_{z\in S} \mathbb{P}_x(X_{n+m}=y \mid X_n=z)\,\mathbb{P}_x(X_n=z)$$

$$= \sum_{z\in S} P^{(m)}_{z,y} \cdot P^{(n)}_{x,z} = (P^n \cdot P^m)_{x,y}$$

The transition matrices P^n can also be seen as acting on the space $\mathcal{P}(S)$ by right multiplication.

Explicitly $P^n : \mathcal{P}(S) \to \mathcal{P}(S)$, $\mu \mapsto \mu P^n$, with

$$\boxed{\mu P^n(y) = \sum_{x\in S} \mu(x)\, P^{(n)}_{x,y}}$$

In particular, from (B)

$$\mu P^n(y) = \sum \mu(x)\cdot \mathbb{P}_x(X_n=y) = \mathbb{P}_\mu(X_n=y)$$

In other words, given that $X_0 \sim \mu \Rightarrow X_n \sim \mu P^n$, $\forall n\in\mathbb{N}$

In short: $\boxed{Law_\mu(X_n) = \mu P^n}$

↳ special case: $\mu = \delta_x \Rightarrow \delta_x P^n = P^{(n)}_{x,\cdot}$ (x-row of P^n)

$$Law_{\delta_x}(X_n) = \delta_x P^n$$

We give an equivalent formulation of the Markov property that can be reinforced to a Strong Markov property.

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Markov Property - equivalent formulation

Let $(X_n)_{n\geq 0}$ be a given stochastic process from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(S, \mathcal{S})$ and let $\mathcal{F}_m^X = \sigma(X_m, \forall m \leq n)$, $n \in \mathbb{N}_0$, its natural filtration.

Then $(X_n)_{n\geq 0}$ has the Markov property $\iff$ $\forall A \in \mathcal{S}$, $\forall n, m \in \mathbb{N}_0$

(a) $\boxed{\mathbb{P}_\mu(X_{n+m} \in A \mid \mathcal{F}_m^X) = \mathbb{P}_\mu(X_{n+m} \in A \mid X_m)}$ $\quad \mathbb{P}_\mu$- a.s.

In particular, if $(X_n)_{n\geq 0}$ is a homogeneous MC

$$\mathbb{P}_\mu(X_{n+m} \in A \mid X_m) = \mathbb{P}_{X_m}(X_n \in A)$$

should be understood as a component of a second MC which starts its dynamics at X_m

In other words, from (a), (b): $Law_\mu((X_{n+m})_{n\geq 0} \mid \mathcal{F}_m^X) = Law_{X_m}((X_n)_{n\geq 0})$

Proof

$\Rightarrow$) is obvious since the cylinder events $\{X_0 = x_0, \dots, X_m = x_m\}$ generate $\mathcal{F}_m^X$, and satisfy the equality.

$\Leftarrow$) Take $A = \{y\}$ and $E = \{X_1 = x_1, \dots, X_m = x\} \in \mathcal{F}_m^X$

From (a) we get

$$\mathbb{P}(X_{n+m} = y, E) = \mathbb{E}\left(\mathbb{E}(\mathbb{1}_{X_{n+m}=y} \mid \mathcal{F}_m^X) \cdot \mathbb{1}_E\right) =$$

$$\overset{(a)}{=} \mathbb{E}\left(\mathbb{P}(X_{n+m} = y \mid X_m) \mathbb{1}_E\right) = \mathbb{P}(X_{n+m} = y \mid X_m = x)\mathbb{P}(E)$$

$\forall \omega \in E: \mathbb{P}(X_{n+m} = y \mid X_m)(\omega) = \mathbb{P}(X_{n+m} = y \mid X_m = x)$

Thus $\mathbb{P}(X_{n+m} = y \mid E) = \mathbb{P}(X_{n+m} = y \mid X_m = x) = \mathbb{P}_x(X_n = y)$ #

Markov property

Theorem (Strong Markov Property)

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Let τ be a stopping time for a MC $(X_n)_{n\geq 0}$, a.s. finite and let $\mathcal{F}_\tau$ the corresponding σ-algebra. Then, $\forall A \in \mathcal{S}$:

$$\boxed{\mathbb{P}_\mu(X_{n+\tau} \in A \mid \mathcal{F}_\tau) = \mathbb{P}_\mu(X_{n+\tau} \in A \mid X_\tau) = \mathbb{P}_{X_\tau}(X_n \in A)}$$

Strong Markov property

Equivalently: $\mathrm{Law}_\mu\left((X_{\tau+n})_{n\geq 0} \mid \mathcal{F}_\tau\right) = \mathrm{Law}_{X_\tau}\left((X_n)_{n\geq 0}\right)$

Proof: Recall that $\mathcal{F}_\tau = \{E \in \mathcal{F} \text{ s.t } E \cap \{\tau \leq n\} \in \mathcal{F}_n, \forall n \geq 0\}$

By the Tower property of conditional expectation, and the Markov property of $(X_n)_{n\geq 0}$, we get:

$$\mathbb{P}_\mu(X_{n+\tau} \in A \mid \mathcal{F}_\tau) = \mathbb{E}_\mu\left(\mathbb{1}_{\{X_{n+\tau} \in A\}} \mid \mathcal{F}_\tau\right)$$

$$= \sum_{m=0}^{\infty} \underbrace{\mathbb{E}_\mu\left(\mathbb{1}_{\{\tau=m\}} \mathbb{1}_{\{X_{n+\tau} \in A\}} \mid \mathcal{F}_\tau\right)}_{\in \mathcal{F}_m}$$

$$= \sum_{m=0}^{\infty} \mathbb{E}_\mu\left(\mathbb{1}_{\{\tau=m\}} \cdot \mathbb{E}_\mu\left(\mathbb{1}_{\{X_{n+m} \in A\}} \mid \mathcal{F}_m\right) \mid \mathcal{F}_\tau\right)$$

|| Markov Property

$$= \sum_{m=0}^{\infty} \mathbb{E}_\mu\left(\mathbb{1}_{\{\tau=m\}} \cdot \mathbb{P}_{X_m}(X_n \in A) \mid \mathcal{F}_\tau\right)$$

$$= \mathbb{P}_{X_\tau}(X_n \in A) \qquad \#$$

Construction of MC

A deterministic dynamics at discrete time is a function $F: S \to S$ s.t., for every initial condition $x \in S$, the whole trajectory $(x_n)_{n\geq 0}$ is recursively defined by $x_{n+1} = F(x_n)$

A MC can be obtained similarly by adding some randomness at each step:

at each step:

Take $(U_n)_{n\geq 1}$ iid $\sim \mathcal{U}[0,1]$ and $F: S\times[0,1]\to S$

Set
$$(1)\begin{cases} \cdot\ X_0\sim\mu \quad \text{independent of } (U_n)_{n\geq 1} \\ \cdot\ X_{n+1} = F(X_n, U_n)\end{cases}$$

This is the content of the following result

$\boxed{\text{Theorem}}$: Let (X_n) be a MC with transition matrix P and initial measure μ. Then, $\exists\, F: S\times[0,1]\to S$ s.t. (X_n) is given by (1).

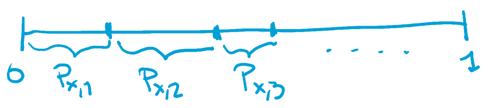
Proof: By definition $X_0\sim\mu$. ✓

We have to construct F from P.

1. For any $x\in S$, consider a partition $I_x = (I_{x,y})_{y\in S}$ of $[0,1]$ s.t. $\lambda(I_{x,y}) = P_{x,y}$, for example identifying S with $\mathbb{N}$ (it is countable)

   λ ↳ Lebesgue on $\mathbb{R}$

   and setting: $I_{x,1} = [0, P_{x,1}]$, $I_{x,2} = [P_{x,1}, P_{x,1}+P_{x,2}]$, ...



Hence, if $(\Omega, \mathcal{F}, \mathbb{P})$ is the space describing X_0 and $(U_n)_{n\in\mathbb{N}}$, we have $\mathbb{P}(U_n\in I_{x,y}) = P_{x,y}$

2. Define, for $x\in S$ and $u\in[0,1]$
$$F(x,u) := \sum_{y\in S} y\cdot \mathbb{1}_{I_{x,y}}(u) = \begin{cases} y & \text{if } u\in I_{x,y} \\ 0 & \text{otherwise}\end{cases}$$

Then, if $X_{n+1} := F(X_n, U_n)$, we have
$$\mathbb{P}(X_{n+1}=y \mid X_n = x) = \mathbb{P}(F(x,U_n)=y) = \mathbb{P}(U_n\in I_{x,y}) = P_{x,y}$$

$\longrightarrow$ The Markov property is verified

→ The Markov property is verified

→ (X_n) is a MC with trans. matrix P #

EXAMPLES

a. Evolution models for the transmission of a genetic trait

General model assumptions
- The gene has 2 possible fenotypes, say A and B (called types)
- No type is favored
- Cells are haploid → type comes from only 1 ancestor
- the population has constant size $N \in \mathbb{N}$

Wright's model

- Consider a (stochastic) dynamics starting from a 0-generation where there are N individuals, each with type A or B.
- At each time $n \in \mathbb{N}$, a new generation born so that any new individual, independently from the others, choose an ancestor uniformly among the previous generation, and make a clone of the gene.

Let X_n = # of individuals of type A in n-generation, $n \in \mathbb{N}$

so that $X_n \in \{0, 1, \dots, N\} = S$, $\forall n \in \mathbb{N}$

Then $(X_n)_{n \in \mathbb{N}_0}$ is a MC on S.

Indeed, for $n \in \mathbb{N}_0$, $j \in \{1, \dots, N\}$, let

$$Y_j^n = \mathbb{1}\{j\text{-th individual in } n\text{-generation has type } A\}$$

so that $Y_j^n \mid X_{n-1} \sim Be\left(\frac{X_{n-1}}{N}\right)$ and $X_n = \sum_{j=1}^{N} Y_j^n$

$$\Rightarrow X_n \mid X_{n-1} \sim Bin\left(N; \frac{X_{n-1}}{N}\right)$$ (only depends on the

$$\Rightarrow X_m \mid X_{m-1} \sim \mathrm{Bin}\left(N; \frac{X_{m-1}}{N}\right)$$ (only depends on the last visited state)

$\Rightarrow (X_m)_{m\in\mathbb{N}_0}$ is a MC with transition probabilities

$$P_{j,k} = \mathbb{P}(X_m = k \mid X_{m-1} = j) = \mathbb{P}\left(\mathrm{Bin}\left(N, \frac{j}{N}\right) = k\right), \text{ for } j,k \in S$$

Remarks:

1. X is a MC on $\{0, \dots N\}$ s.t. $P_{j,k} > 0 \quad \forall j \in \{1, \dots, N-1\}, k \in S$
   while $P_{0,0} = 1$ and $P_{N,N} = 1$ (hence $P_{0,j} = 0 \ \forall j \neq 0$, $P_{N,j} = 0 \ \forall j \neq N$).
   When this happens, we call the state absorbing, as the MC that ends in it, stays there forever.
   Notice that states 0 and N correspond to the situation in which a type has dominated the other ($0 \to$ only type B, $N \to$ only type A)
   ↳ called fixation

2. The Wright's model can be equivalently defined as the evolution of the frequence of type A among the generations (this is indeed the most common representation):
   $$\forall m \in \mathbb{N}_0 : \quad Z_m := \frac{X_m}{N} \in \left\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\right\} = S'$$
   Then (easy check) $(Z_m)_{m\geq 0}$ is a MC with trans. matrix
   $$P_{x,y} = \mathbb{P}(\mathrm{Bin}(N, x) = N\cdot y), \quad \text{for } x,y \in S'$$
   This representation is convenient because it allows to consider large value of N (hence the limit $N \to \infty$) keeping S' compact.

We state a general result of martingale theory, which is powerful and that will be applied in our context to compute some relevant quantities.

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Theorem (Martingale convergence theorem)

Let $(X_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ - martingale

- If $(X_n)_{n\geq 0}$ is L^1 - bounded (i.e. $\mathbb{E}(|X_n|)\leq c$, $\forall n\geq 0$)

$$\Rightarrow \exists X_\infty \text{, with } \mathbb{E}(X_\infty) < \infty \text{ s.t. } X_n \xrightarrow[n\to\infty]{a.s.} X_\infty$$

- If $(X_n)_{n\geq 0}$ is L^p - bounded for some $p>1$ $\left(\mathbb{E}(|X_n|^p)\leq c \;\; \forall n\geq 0 \right)$

$$\Rightarrow \exists X_\infty \text{ s.t. } X_n \xrightarrow[n\to\infty]{L^p, a.s.} X_\infty$$

- Probability of fixation

From the definition, it turns that $(X_n)_{n\in\mathbb{N}_0}$ is a bounded martingale. Indeed, $\forall j\in S$ initial state $(X_0\sim\delta_j)$:

$$\mathbb{E}_j(X_{n+1}|X_n) = \mathbb{E}\left(\mathrm{Bin}\left(N, \frac{X_n}{N}\right)\Big|X_n\right) = X_n \qquad \forall n\in\mathbb{N}_0$$

so that $\mathbb{E}_j(X_n) = \mathbb{E}_j(X_0) = j$

From the martingale convergence theorem:

$$X_n \xrightarrow[n\to\infty]{} X_\infty \qquad \mathbb{P}_j\text{ - a.s. and in } L^2,$$

with $\mathbb{E}_j(X_\infty) = \mathbb{E}_j(X_0) = j$

* If we can argue (as it is intuitive) that $X_\infty\in\{0,N\}$, then $j = \mathbb{E}_j(X_\infty) = N\cdot\mathbb{P}(X_\infty = N)$

$$\Rightarrow \mathbb{P}(X_\infty = N) = \frac{j}{N} \quad\text{and}\quad \mathbb{P}(X_\infty = 0) = 1-\frac{j}{N}$$

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b. Evolution of state (e.g. opinion) among interacting individuals

↳ interacting particle systems

(- individuals (or particles) interact through a

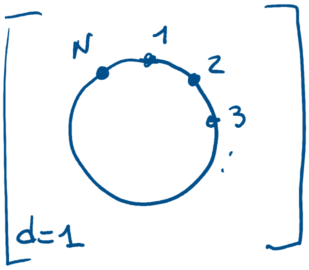
- Interacting particle systems

General model assumptions

- individuals (or particles) interact through a "network of friends" that is described by a graph $G=(V,E)$, where vertices are the position of individuals and edges correspond to friendship among a couple
- the state (or opinion) takes value on finite E so that the whole system is described by states in E^G → state or configuration space
- The dynamics change the state of a single individual at each step, and only depends on the state of its neighborhood.

Voter model

- As a graph of interaction, consider $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$, and assume that there is a votant at any vertex of $\mathbb{T}_N^d$ (hence N^d in total)



- Any votant holds an opinion $\in \{0,1\}$ (favorable / against) so that the whole system of opinions takes value in $S = \{0,1\}^{\mathbb{T}_N^d} \ni x = (x(i))_{i\in\mathbb{T}_N^d}$ and $x(i) \in \{0,1\}$
- At each time $n \in \mathbb{N}$, an individual, chosen uniformly in $\mathbb{T}_N^d$, choose uniformely one of its neighbors and agree with its opinion. Hence the whole set of opinions has changed.

Let X_n = set of opinions after n steps $= (X_n(i))_{i\in\mathbb{T}_N^d}$, $\forall n \in \mathbb{N}$

so that $X_n(i)$ = opinion of votant at site i, $i \in \mathbb{T}_N^d$

Then $(X_n)_{n\in\mathbb{N}_0}$ is a MC on S.

Indeed: let $(I_n)_{n\in\mathbb{N}}$ iid $\sim \mathcal{U}(\mathbb{T}_N^d)$

and $(U_n)_{n\in\mathbb{N}}$ iid $\sim \mathcal{U}(\underbrace{\pm e_1, \dots, \pm e_d}_{\text{neighborhood in } \mathbb{T}_N^d})$ with $e_k = (0\dots 0, \underset{k\text{-comp.}}{1}, 0 \dots 0)$

neighborhood in Π_N^d k-comp.

Then $(X_n)_{n\geq 0}$ is recursively defined so that

$$(*) \qquad X_n(i) = \begin{cases} X_{n-1}(i) & \text{if } I_n \neq i \\ X_{n-1}(\underbrace{I_n + U_n}_{\text{chosen neighbor of } I_n}) & \text{if } I_n = i \end{cases}$$

which is clearly a MC with transition prob. that can be obtained from $(*)$ (try as an exercise).

Remark:

Since the evolution rule "made copy of other opinions", as for the Wright's model there are 2 absorbing states which correspond to the "total agreement", namely the states:

$\underline{0} = \{$all votants have opinion $0\}$ and $\underline{1} = \{$all votants have opinion $1\}$

- Probability of total agreement

One can consider the sequence $(M_n)_{n\geq 0}$, where

$$M_n = \sum_{i \in \Pi_N^d} X_n(i) = \# \text{ votants with opinion } 1 \in \{0, \dots, N^d\}$$

Then $(M_n)_{n\geq 0}$ is a sequence of r.v.'s on $\{0, \dots, N^d\}$

In particular it is a martingale (bounded in L^p, $\forall p \geq 1$) w.r.t. $\mathcal{F}_n = \sigma(X_k, k \leq n)$:

$$\mathbb{E}(M_n \mid \mathcal{F}_{n-1}) = M_{n-1} - \mathbb{E}(X_{n-1}(I_n) \mid \mathcal{F}_{n-1}) + \mathbb{E}(X_{n-1}(I_n + U_n) \mid \mathcal{F}_{n-1})$$

$$= M_{n-1} - \sum_{i \in \Pi_N^d} X_{n-1}(i) \cdot \underset{=\frac{1}{N^d}}{\cancel{\mathbb{P}(I_n = i)}} + \sum_{i \in \Pi_N^d} X_{n-1}(i) \underset{=\frac{1}{N^d}}{\cancel{\mathbb{P}(I_n + U_n = i)}}$$

$$= M_{n-1}$$

$= M_{n-1}$

By the martingale convergence theorem we then have

$$M_n \xrightarrow[n\to\infty]{} M_\infty \quad \mathbb{P}_j\text{-a.s. and in } L^2, \ \forall j \text{ initial state}$$

so that $\quad \mathbb{E}_j(M_\infty) = \mathbb{E}_j(M_0) = j$

As before, if we can argue that $M_\infty \in \{0, N^d\}$

$\Longrightarrow \quad \mathbb{P}_j(M_\infty = N^d) = \frac{j}{N^d}$ and $\mathbb{P}_j(M_\infty = 0) = 1 - \frac{j}{N^d}$
