# ADVANCED STOCHASTIC PROCESSES - 1° LECTURE

## INTRODUCTION

- The course deals with stochastic processes $X = (X_t)_{t \in I}$, defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking value on a measurable space $(S, \mathcal{S})$, where

$$\begin{cases} \cdot\ I = \mathbb{N}_0 \ (n \geq 0) \quad \text{or} \quad \mathbb{R}^+ \ (t \geq 0) \quad (\text{or subsets, } [0,T]) \\ \cdot\ S = \text{countable space} \quad \text{or} \quad \text{uncountable (metric) space (e.g. } \mathbb{R} \text{ or } \mathbb{R}^d) \end{cases}$$

On the whole $X : \Omega \longrightarrow S^I = \{x : I \longrightarrow S\}$ set of trajectories

$$\omega \longrightarrow X(\omega) : I \longrightarrow S$$
$$t \longrightarrow X_t(\omega)$$

<u>Remark</u>: $S^I$ is the product space endowed by the product $\sigma$-algebra $\mathcal{S}^I \equiv \sigma(\mathcal{C})$, with $\mathcal{C} = \{\text{cylinder sets of } S^I\}$.

Recall: $\forall\, k \in \mathbb{N}$, $\underline{t} = (t_1, \dots, t_k)$ with $t_j \in I$, $\underline{A} = (A_1, \dots, A_k)$ with $A_j \in \mathcal{S}$

cylinder set $C(\underline{t}, \underline{A}) = \{x \in S^I : x_{t_1} \in A_1, \dots, x_{t_k} \in A_k\}$

Let's also recall that the natural filtration of $X$ on $(\Omega, \mathcal{F})$:

$$\mathcal{F}_t^X := \sigma(X_s, s \leq t), \quad \forall\, t \in I.$$

- The focus is mainly on Markov Processes $X = (X_t)_{t \in I}$

<u>General idea</u>:

- $X_t$ depends on 'its history in $[0,s]$' $(\sigma(X_u, u \leq s))$ only through $X_s$ $\longrightarrow$ Markov property
- They are characterized by <u>transition probabilities</u> to go

- They are characterized by transition probabilities to go from $x$ at time $s$ to $y$ at time $t$ $\longrightarrow p(x,s;y,t)$
- We will only consider homogeneous Markov Processes, where these probabilities only depend on $t-s \longrightarrow p(x,s;y,t) = p(x,y,t-s)$

Examples:
- Random walk (for time in $\mathbb{N}_0$)
- Brownian motion (for time in $\mathbb{R}^+$)
- Solution of Stochastic Diff. Eq. (under proper assumptions)

If $S$ is at most countable, we rather speak of Markov Chains (MC)
$\begin{cases} \text{- in discrete times, if } I = \mathbb{N}_0 \\ \text{- in continuous time, if } I = \mathbb{R}^+ \text{ (or subsets)} \end{cases}$

while if $S$ is uncountable, we turn to the general theory of MP and assume $S$ to be a metric space, $(S,d)$, with good properties (metric, complete, separable $\to$ Polish space)

## Organization

- Markov chains
  - ↳ Branching Processes
- Poisson Processes and Poisson point measure
- Markov chains in continuous time
  - ↳ interacting particle systems
- Markov Processes and Feller generators
  - ↳ SDE and connection to PDE
- Infinite-divisible laws and Lévy Processes

- Convergence of measures (laws of processes)

  ↳ Invariance Principle

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## Markov chains (MC)

- Let $S$ be a countable space and let

$$\mathcal{P}(S) := \{\text{probability measures on } S\}$$
$$= \left\{ (\nu(x))_{x\in S} : \nu(x) \geq 0, \sum_{x\in S} \nu(x) = 1 \right\}$$

$\boxed{\text{Def.}}$ A transition probability on $S$ is a function

$$p : S \times S \to [0,1] \quad \text{s.t.} \quad \begin{cases} P_{x,y} \geq 0 & \forall x,y \in S \\ \sum_{y\in S} P_{x,y} = 1 & \forall x \in S \end{cases}$$
$$(x,y) \to P_{x,y}$$

The function $p$ can be identified with a matrix $P = (P_{x,y})_{x,y\in S}$ called transition matrix.

Remark:

- $P_{x,y}$ will represent the probability of the transition from $x$ to $y$ in one step (or in one unit of time).
- $\forall x \in S$, $P_{x,\cdot} : S \to [0,1]$, $y \to P_{x,y}$ is a density on $S$ $\left( P_{x,\cdot} \in \mathcal{P}(S) \right)$

$\boxed{\text{Def}}$ The sequence of r.v. $(X_n)_{n\in\mathbb{N}_0}$ with value on $S$ is a Markov chain with transition matrix $P$ if,

for all events $\{\underbrace{X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_n = x}_{\in \mathcal{F}_n^X}\}$ of non-zero probability:

cylinder event

$$\mathbb{P}(X_{n+1} = y \mid X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_n = x) = \mathbb{P}(X_{n+1} = y \mid X_n = x) = \mathbb{P}(X_1 = y \mid X_0 = x) = P_{x,y}$$

$$\mathbb{P}(X_{m+1}=y \mid X_0=x_0,\dots,X_{m-1}=x_{m-1}, X_m=x) \underset{\text{Markov property}}{=} \mathbb{P}(X_{m+1}=y \mid X_m=x) \underset{\text{homogeneity}}{=} \mathbb{P}(X_1=y \mid X_0=x) = P_{x,y}$$

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<u>Example</u>: Let $(\xi_m)_{m\in\mathbb{N}}$ iid r.v. taking value on $\mathbb{Z}$ with density $(\nu(k))_{k\in\mathbb{Z}}$ $\left(\mathbb{P}(\xi_m=k)=\nu(k)\right)$, and define $(S_m)_{m\in\mathbb{N}_0}$: $S_0=0$, $S_m=\sum_{k=1}^{m}\xi_k$

$\to$ $(S_m)_{m\in\mathbb{N}_0}$ is called <u>random walk on $\mathbb{Z}$</u> (with i.i.d. steps or increments)

From definition, it turns that

$$\mathbb{P}(\overset{S_m+\xi_{m+1}}{\overset{\shortparallel}{S_{m+1}}}=k \mid \underbrace{S_0=0, S_1=j_1,\dots,S_m=j}_{\text{trajectory with prob} \neq 0}) = \mathbb{P}(S_{m+1}=k \mid S_m=j) = \mathbb{P}(\xi_{m+1}=k-j) = \nu(k-j)$$

Then $(S_m)_{m\geq 0}$ is a MC with trans. probabilities $P_{j,k}=\nu(k-j)$

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We can look at $X=(X_m)_{m\in\mathbb{N}_0}$ as a whole random trajectory:

$$X: \Omega \longrightarrow S^{\mathbb{N}_0}$$
$$\omega \longrightarrow X(\omega)=(X_m(\omega))_{m\in\mathbb{N}_0},$$

whose law is identified by its value on the cylinder events of type: $\{X_0=x_0,\dots,X_m=x_m\}$, $\forall\, m\in\mathbb{N}$, $\forall\, x_0,\dots,x_m\in S$

As for deterministic dynamics, the trajectory (or rather its law) is identified with the additional choice of the "initial state". For stochastic processes, this corresponds to the choice of an <u>initial measure $\mu\in\mathcal{P}(S)$</u>, so that <u>$X_0\sim\mu$</u> ($X_0$ has law $\mu$)

The law of a MC $(X_m)_{m\geq 0}$ with initial measure $\mu$ and transition

The law of a MC $(X_n)_{n\geq 0}$ with initial measure $\mu$ and transition matrix $P=(p_{x,y})_{x,y\in S}$ is denoted by $\mathbb{P}_\mu$ and satisfies, $\forall n\in\mathbb{N}_0$

$$(A)\quad \mathbb{P}_\mu\left(X_0=x_0,\dots,X_n=x_n\right) \overset{\substack{\text{recursively use Markov property}\\ \downarrow}}{=} \mu(x_0)\,p_{x_0,x_1}\,p_{x_1,x_2}\cdots p_{x_{n-1},x_n} = \mu(x_0)\prod_{k=0}^{n-1} p_{x_k,x_{k+1}}$$

$$\| \qquad\qquad \| $$

$$\underbrace{\mathbb{P}_\mu\left(X_n=x_n \mid \cancel{X_0=x_0,\dots,}\,X_{n-1}=x_{n-1}\right)}_{=\,p_{x_{n-1},x_n}}\cdot \mathbb{P}_\mu\left(X_0=x_0,\dots,X_{n-1}=x_{n-1}\right) = \text{iterations} = \dots$$

Remarks:

$\boxed{1}$ there is some ambiguity in calling $\mathbb{P}_\mu$ law of $X$, as it is rather a measure over $\Omega$. However the ambiguity disappears if we consider the canonical representation of the process:

- $\Omega = S^{\mathbb{N}_0} \ni x = (x_n)_{n\in\mathbb{N}_0}$ and set $X_n(x) := x_n$, $\forall n\in\mathbb{N}_0$

$\boxed{2}$ The identity (A) indeed implies that $X_0 \sim \mu$ (obvious) and that $(X_n)$ is a MC - satisfies the Markov property - (easy to verify), then is an alternative definition of MC.