Advanced Stochastic Processes

Master's course in Mathematics 2025-2026

Final exercise sheet

Exercise 1. Let P be an irreducible transition matrix on the countable state space S, with stationary distribution π. Let $A \subset S$ and define the truncation of P on A to be the transition matrix $\hat{P} = (\hat{p}_{i,j})_{i,j \in A}$ given by

$$\hat{p}_{i,j} = \begin{cases} p_{i,j} & \text{if } i \neq j \\ p_{i,i} + \sum_{k \in A^c} p_{i,k} & \text{if } i = j \end{cases} , \ \forall i, j \in A$$

Show that if P is reversible w.r.t π, then so is $\hat{P}$ w.r.t. $\frac{\pi \cdot \mathbb{1}_A}{\pi(A)}$.

Exercise 2. Let $X = (X_n)_{n \geq 0}$ be a (homogeneous) irreducible Markov chain with state space S and transition matrix P. Define the sequence $(\tau_n)_{n \geq 0}$ recursively by $\tau_0 = 0$ and for $n \geq 0$

$$\tau_{n+1} := \inf\{k \geq \tau_n + 1 \, : \, X_k \neq X_{\tau_n}\} ,$$

and then define $Y_n := X_{\tau_n}$, for all $n \geq 0$. Show that $(Y_n)_{n \geq 0}$ is a homogeneous Markov chain and give its transition matrix.

Exercise 3. Let $X = (X_n)_{n \geq 0}$ be a (homogeneous) irriducible and positive recurrent Markov chain with state space S and transition matrix P. Show that, for all $i, j \in S$ and any initial distribution ν,

$$\frac{\sum_{k=1}^{n} \mathbb{1}_{\{X_k = i\}} \mathbb{1}_{\{X_{k+1} = j\}}}{\sum_{k=1}^{n} \mathbb{1}_{\{X_k = i\}}} \underset{n \to \infty}{\longrightarrow} p_{i,j} , \quad \mathbb{P}_\nu - a.s.$$

Exercise 4. Let $X = (X_n)_{n \geq 0}$ be a homogeneous Markov chain with countable state space S and transition matrix P. Let D be a subset of S such that $\mathbb{P}_x(\tau < \infty) = 1$ for any $x \in S$, where $\tau := \inf\{n \geq 0 : \, X_n \notin D\}$ is the hitting time of $D^c = S \setminus D$, and consider two nonnegative real functions $c : D \mapsto \mathbb{R}^+$ (*unit cost function*) and $f : D^c \mapsto \mathbb{R}^+$ (*final cost function*).

a. Verify that the function $h : S \mapsto \mathbb{R}^+$, given by

$$h(x) := \mathbb{E}_x \left[\sum_{k=0}^{\tau - 1} c(X_k) + f(X_\tau) \right] ,$$

satisfies the following problem

$$(P) \qquad \begin{cases} Ph(x) - h(x) = -c(x) & \text{if } x \in D \\ h(x) = f(x) & \text{if } x \in D^c \end{cases} .$$

b. Show that h (if bounded) is the unique possible bounded solution of (P).

Exercise 5. A countable number of particles move independently in the countable space S, each according to a Markov chain with transition matrix P. Let $A_n(i)$ be the number of particles in state $i \in S$ at time $n \geq 0$, and suppose that the $(A_0(i))_{i\in S}$ are independent Poisson random variables with respective means $\mu(i)$, $i \in S$, where $\mu = (\mu(i))_{i\in S}$ is an invariant measure of P. Show that for all $n \geq 1$, the random variables $(A_n(i))_{i\in S}$ are independent Poisson random variables with respective means $\mu(i)$, $i \in S$.

Exercise 6. Let $N = (N_t)_{t\geq 0}$ be a homogeneous Poisson process (on $\mathbb{R}^+$) with intensity $\lambda > 0$. For given $0 < s < t$, prove that for all $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$

$$\mathbb{P}\left(N_s = k \,|\, N_t = n\right) = \binom{n}{k} p_{s,t}^k (1 - p_{s,t})^{n-k}\,, \qquad \text{with } p_{s,t} = \frac{s}{t}\,.$$

Exercise 7. Let N be a Poisson point process on $\mathbb{R}^2$ with mean $\mu = cLeb$, where $c > 0$ is a finite constant, and let R be the (random) distance from the origin of the nearest point. Compute the distribution of R on $\mathbb{R}^+$.

Exercise 8. Let Λ be a finite measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, N be a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^+$ with intensity measure $Leb \times \Lambda$, and let $(X_t)_{t\geq 0}$ denote the compound Poisson process on $\mathbb{R}^+$ given by

$$X_t := \int_0^t \int_0^{+\infty} x N(ds, dx)\,, \quad t \geq 0\,.$$

(a) Prove that $(X_t)_{t\geq 0}$ has stationary and independent increments;

(b) Show that the Laplace transform of X_t, namely $\mathbb{E}(e^{-\theta X_t})$, with $\theta > 0$, corresponds to the Laplace functional of N for a suitable function f, and then compute its value;

(c) Assume that $\int_{\mathbb{R}^+} x^2 \Lambda(dx) < \infty$, and compute the average and the variance of X_t.

Exercise 9. Consider the uniform Markov chain $(X_t)_{t\geq 0}$ with state space $S = \{0, 1\}$, and such that the subordinated chain has transition matrix

$$\hat{P} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}\,, \text{ for } \alpha, \beta \in (0,1),$$

and the underlying Poisson clock has intensity $\lambda > 0$.

a. Find the transition semigroup $(P_t)_{t\geq 0}$.

b. Suppose that $X_0 = 0$. Give the joint probability distribution of the sequence $\{\tau_n - \tau_{n-1}\}_{n\geq 1}$, where $\tau_1, \tau_2, \dots$ are the successive times when X switches from one value to a different value.

Exercise 10. Let $N = (N_t)_{t\geq 0}$ be a homogeneous Poisson process with intensity $\lambda > 0$, and define the process $(X_t)_{t\geq 0}$, taking value on $S = \{-1, +1\}$, such that

$$X_t := X_0 \cdot (-1)^{N_t}$$

where X_0 is a random variable with value in S, independent of N.

a. Verify that X is a continuous-time Markov chain and compute its transition probabilities.

b. Compute the generator of X and its invariant distribution.

Exercise 11. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion taking values on $\mathbb{R}^d$, with $B_t = (B_t^1, B_t^2, \dots, B_t^d)$. Consider the corresponding Bessel process, $(R_t)_{t\geq 0}$, defined by

$$R_t := |B_t| = \sqrt{\sum_{j=1}^{d} (B_t^j)^2}\,, \quad \forall t \geq 0\,,$$

and compute its generator L.

Exercise 12. Let $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto M_{d\times d}$ be Lipschitz continuous functions. For $x \in \mathbb{R}^d$, and given a standard Brownian motion $(B_t)_{t\geq 0}$ with value in $\mathbb{R}^d$, let $X = (X_t)_{t\geq 0}$ be the unique strong solution of SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x \end{cases}$$

Denoting by x^i the i-th component of $x \in \mathbb{R}^d$, for $i = 1, \dots d$, prove that for all $i, j \in \{1, \dots, d\}$

$$\lim_{t\downarrow 0} \frac{1}{t}\left[\mathbb{E}_x(X_t^i) - x^i\right] = b^i(x)\,, \quad \lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_x\left[(X_t^i - x^i)(X_t^j - x^j)\right] = (\sigma\sigma^*)_{ij}\,.$$

Exercise 13. Let D be an open and bounded domain of $\mathbb{R}^d$ with boundary ∂D, and consider two real bounded functions $g : D \mapsto \mathbb{R}$ and $f : \mathbb{R}^d \mapsto \mathbb{R}$. In this setting, a solution of the Schördinger problem is a function $u : \bar{D} \mapsto \bar{\mathbb{R}}$ that satisfies the following

$$(S) \qquad \begin{cases} \frac{1}{2}\Delta u(x) = -g(x)u(x) & \text{if } x \in D \\ u(x) = f(x) & \text{if } x \in \partial D \end{cases}.$$

Show that if the above problem admits a bounded solution, then it should be given by

$$\psi(x) := \mathbb{E}_x\left[\exp\left(\int_0^\tau g(B_s)ds\right) f(B_\tau)\right]\,,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$ and τ is the corresponding hitting time on ∂D, namely $\tau := \inf\{t > 0 \,:\, B_t \notin D\}$.
Hint: Given a bounded solution u of (S), let $M_t := u(B_t)\exp\left(\int_0^t g(B_s)ds\right)$, and show that, for $t < \tau$, it is a local martingale.