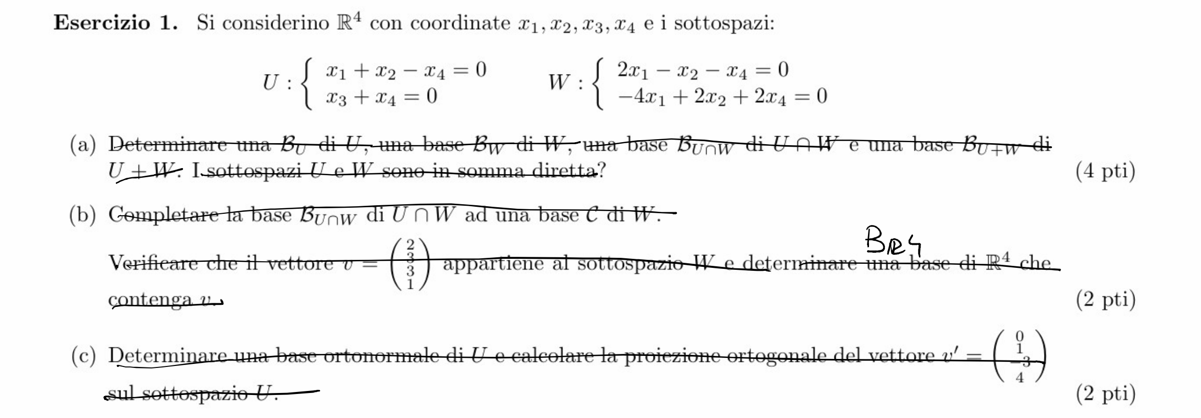
Lezione38



Esercizio 1. Si considerino $\mathbb{R}^4$ con coordinate x_1, x_2, x_3, x_4 e i sottospazi:

$$U : \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases} \qquad W : \begin{cases} 2x_1 - x_2 - x_4 = 0 \\ -4x_1 + 2x_2 + 2x_4 = 0 \end{cases}$$

(a) ~~Determinare una $\mathcal{B}_U$ di U, una base $\mathcal{B}_W$ di W, una base $\mathcal{B}_{U\cap W}$ di $U \cap W$ e una base $\mathcal{B}_{U+W}$ di $U + W$. I sottospazi U e W sono in somma diretta~~? (4 pti)

(b) ~~Completare la base $\mathcal{B}_{U\cap W}$ di $U \cap W$ ad una base $\mathcal{C}$ di W.~~

~~Verificare che il vettore $v = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix}$ appartiene al sottospazio W e determinare una base di $\mathbb{R}^4$ che contenga v.~~ $B_{\mathbb{R}^4}$ (2 pti)

(c) ~~Determinare una base ortonormale di U e calcolare la proiezione ortogonale del vettore $v' = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 4 \end{pmatrix}$ sul sottospazio U.~~ (2 pti)

Svolgimento

a) $U : \begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$ $\dim U = 4-2 = 2$

$$\begin{cases} \boxed{x_1} = -x_2 + x_4 \\ \boxed{x_3} = -x_4 \end{cases} \qquad \begin{pmatrix} \boxed{x_1} \\ x_2 \\ \boxed{x_3} \\ x_4 \end{pmatrix} \qquad U = \left\{ \begin{pmatrix} -x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

$$\boxed{B_U = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\},\ \dim U = 2}$$

$W : \begin{cases} 2x_1 - x_2 - x_4 = 0 \\ \cancel{-4x_1 + 2x_2 + 2x_4 = 0} \end{cases}$ $\dim W = 4 - 1 = 3$

$$\boxed{x_4} = 2x_1 - x_2 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \boxed{x_4} \end{pmatrix} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 2x_1 - x_2 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\boxed{B_W = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\},\ \dim W = 3}$$

Essendo $\dim U = 2$ $\dim W = 3$ se U fosse in somma diretta con W

$\dim(U \oplus W) = 2 + 3 = 5$ ma $U \oplus W \leq \mathbb{R}^4$ impossibile

$5 \leq 4$

$\boxed{\text{Non sono in somma diretta}}$

Calcoliamo $U \cap W$:

$$\begin{cases} x_1 + x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \\ 2x_1 - x_2 - x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = -x_2 + x_4 \\ x_3 = -x_4 \\ -2x_2 + 2x_4 - x_2 - x_4 = 0 \end{cases}$$

$$\begin{cases} x_1 = -x_2 + x_4 \\ x_3 = -x_4 \\ -3x_2 + x_4 = 0 \end{cases} \qquad \begin{cases} \enclose{circle}{x_1} = -x_2 + 3x_2 = 2x_2 \\ \enclose{circle}{x_3} = -3x_2 \\ \enclose{circle}{x_4} = 3x_2 \end{cases} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \left\{ \begin{pmatrix} 2x_2 \\ x_2 \\ -3x_2 \\ 3x_2 \end{pmatrix} \middle| \, x_2 \in \mathbb{R} \right\}$$

$$U \cap W = \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix} \right\rangle \qquad \boxed{B_{U\cap W} = \left\{ \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix} \right\}, \ \dim(U \cap W) = 1}$$

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W) = 2 + 3 - 1 = 4$$

$$\Rightarrow \ U+W \subseteq \mathbb{R}^4 \qquad \dim(U+W) = 4 \Longrightarrow U+W = \mathbb{R}^4$$

$$\boxed{B_{U+W} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \ \dim(U+W) = 4}$$

b) Completare $B_{U\cap W}$ ad una base $\mathcal{C}$ di W.

$$B_{U\cap W} = \left\{ \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix}, \underline{w}, w' \right\} \text{ sia base di } W$$

$$\boxed{w \in W} \qquad w \notin \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix} \right\rangle$$

$$B_W = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad w = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$w' \in W \qquad w' \notin \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 2a+b \\ a \\ -3a \\ 3a+2b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad \enclose{circle}{\begin{matrix} a = 1 \\ -3a = 0 \end{matrix}} \ \text{imp.}$$

$$\boxed{\begin{matrix} \mathcal{C} = \left\{ \underset{U\cap W}{\begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix}}, \underset{\substack{w \\ w \in W}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}}, \underset{w' \in W}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}} \right\} \\ w \notin \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix} \right\rangle \qquad w' \notin \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\rangle \end{matrix}}$$

Verifichiamo che $v = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix} \in W : \{ 2x_1 - x_2 - x_4 = 0$

$$2 \cdot 2 - 3 - 1 = 4 - 3 - 1 = 0 \ ✓ \ \Rightarrow v \in W.$$

$$W = \langle w_1, w_2, w_3 \rangle \qquad \operatorname{rg} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ v \end{pmatrix} = 3 \iff v \in W$$

$\dim W = 3$

$$B_{\mathbb{R}^4} = \left\{ \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \operatorname{rg} \begin{pmatrix} 2 & 3 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4$$

c) Determiamo una base $\overline{B}=\{u_1,u_2\}$ ortonormale di U usando G-S

$$\left\{\underset{v_1}{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}},\underset{v_2}{\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}}\right\}$$

$$u_1=\frac{v_1}{\|v_1\|}=\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}$$

$$w_2=v_2-(v_2\cdot u_1)\,u_1=\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}-\Bigg[\underbrace{\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}}_{-\frac{1}{\sqrt{2}}}\Bigg]\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}-\begin{pmatrix}1/2\\-1/2\\0\\0\end{pmatrix}=\begin{pmatrix}1/2\\1/2\\-1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\\-2\\2\end{pmatrix}$$

$$u_2=\frac{w_2}{\|w_2\|}=\frac{\cancel{2}}{\sqrt{10}}\,\frac{1}{\cancel{2}}\begin{pmatrix}1\\1\\-2\\2\end{pmatrix}=\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix}$$

$$\|w_2\|=\left\|\frac{1}{2}\begin{pmatrix}1\\1\\-2\\2\end{pmatrix}\right\|=\frac{1}{2}\sqrt{1+1+4+4}=\frac{\sqrt{10}}{2}$$

$$\boxed{\overline{B}=\left\{\overset{u_1}{\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}},\overset{u_2}{\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix}}\right\}}\qquad \overline{B}=\{u_1,u_2\}$$

Calcoliamo $P_U(v')$ $\quad v'=\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}$

$$\boxed{P_U(v')=(v'\cdot u_1)u_1+(v'\cdot u_2)v_2}$$

$$P_U\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}=\Bigg[\overset{1/\sqrt{2}}{\underline{\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}}}\Bigg]\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}+\Bigg[\underbrace{\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix}}_{\frac{1+6+8}{\sqrt{10}}}\Bigg]\begin{pmatrix}1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\\2/\sqrt{10}\end{pmatrix}=$$

$$=\begin{pmatrix}-1/2\\1/2\\0\\0\end{pmatrix}+\begin{pmatrix}\cancel{15}^{3}/\cancel{10}_{2}\\\cancel{15}^{3}/\cancel{10}_{2}\\-\cancel{30/10}^{-3}\\\cancel{30/10}^{3}\end{pmatrix}=\begin{pmatrix}1\\2\\-3\\3\end{pmatrix}$$

$$\boxed{P_U\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}=\begin{pmatrix}1\\2\\-3\\3\end{pmatrix}}$$

$$v'=\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}=v'_{\parallel}+v'_{\perp}\qquad \begin{matrix}v'_{\parallel}\in U\\v'_{\perp}\in U^{\perp}\end{matrix}\qquad U:\begin{cases}x_1+x_2-x_4=0\\x_3+x_4=0\end{cases}$$

$$U^{\perp}=\left\langle\underset{a}{\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}},\underset{b}{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}}\right\rangle$$

$$\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}=v'_{\parallel}+\begin{pmatrix}a\\a\\b\\-a+b\end{pmatrix}$$

$$v'_{\parallel}=\begin{pmatrix}0\\1\\-3\\4\end{pmatrix}-\begin{pmatrix}a\\a\\b\\-a+b\end{pmatrix}=\begin{pmatrix}-a\\1-a\\-3-b\\4+a-b\end{pmatrix}\in U\qquad\begin{cases}x_1+x_2-x_4=0\\x_3+x_4=0\end{cases}\qquad\begin{cases}-a+\underset{\times}{1}-a-\underset{\times}{4}-a+b=0\\-3-b+4+a-b=0\end{cases}$$

$$\begin{cases} -3a+b-3=0 \\ a-2b+1=0 \end{cases} \qquad \begin{cases} b=3a+3 \\ a=2b-1=6a+6-1 \end{cases} \qquad \begin{cases} b=0 \\ a=-1 \end{cases}$$

$U \oplus W = \mathbb{R}^4$

$U \oplus U^{\perp} = \mathbb{R}^4$

$v_{\perp} = (v \cdot \bar{v})\bar{v}$

$$\boxed{v'_{//} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 3 \end{pmatrix}}$$

$P_W(v) = v_{//}$

$\dim W = 3$ in $\mathbb{R}^4$

$\dim W^{\perp} = 1$ $\quad B_{W^{\perp}} = \{\bar{v}\}$ versore di $W^{\perp}$

$$v = v_{//} + v_{\perp} \Rightarrow \boxed{v_{//} = v - v_{\perp} = v - (v \cdot \bar{v})\bar{v}}$$

Esercizio 2. Si consideri l'applicazione lineare $f : \mathbb{R}^3 \to \mathbb{R}^4$ definita da

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x-z \\ 2x-y-3z \end{pmatrix}$$

i) ~~Determinare una base $B_{\mathrm{Ker}(f)}$ di $\mathrm{Ker}(f)$ e una base $B_{\mathrm{Im}(f)}$ di Im~~(f). ~~L'applicazione f è iniettiva; è suriettiva; è biiettiva?~~ (2 pti)

ii) Determinare il rango di A. Calcolare $f^{-1}\left\{\begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix}\right\}$. (1,5 pti)

iii) Si dimostri che la matrice

$$B = \begin{pmatrix} 2 & -4 & 5 \\ -2 & 4 & 1 \\ 0 & 0 & 12 \end{pmatrix}$$

è diagonalizzabile, si determini il polinomio caratteristico di B, gli autovalori di B e i relativi autospazi. (2,5 pti)

iv) Dimostrare che la matrice

$$C = \begin{pmatrix} 0 & 1 & -4 \\ 0 & 12 & 21 \\ 0 & 0 & 6 \end{pmatrix}$$

è simile alla matrice B. (1 pto)

Svolgimento: $\quad f: \mathbb{R}^3 \to \mathbb{R}^4$

i)

$$A_{\mathcal{E}_3,\mathcal{E}_4,f} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -3 \end{pmatrix} = A \qquad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x-z \\ 2x-y-3z \end{pmatrix}$$

Essendo $\dim \mathbb{R}^3 = 3$ del dominio $\Rightarrow$ $\dim V = \dim \mathrm{Ker} f + \dim \mathrm{Im} f$

$3 = 0 + ③$

$1 + 2$

$2 + 1$

$3 + 0$

$\Rightarrow \dim \mathrm{Im} f \leq 3$ perciò

$\dim \mathrm{Im} f \neq 4$ quindi $\boxed{f \text{ non è suriettiva} \Rightarrow \text{non è biiettiva}}$

$B_{\mathrm{Ker} f}$ $\qquad \mathrm{Ker} f = \left\{ v \in \mathbb{R}^3 \mid f(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$\begin{cases} x+y=0 \\ y+z=0 \\ x-z=0 \\ 2x-y-3z=0 \end{cases} \qquad \begin{matrix} -1+1=0 \checkmark \\ 1-1=0 \checkmark \\ -1+1=0 \checkmark \\ -2-1+3=0 \checkmark \end{matrix}$$

$$\begin{cases} x=-y \\ z=-y \\ \cancel{x=z=-y} \\ -2y-y+3y=0 \end{cases} \qquad \begin{cases} ⓧ=-y \\ ⓩ=-y \end{cases} \qquad \begin{pmatrix} ⓧ \\ y \\ ⓩ \end{pmatrix}$$

$$\mathrm{Ker} f = \left\{ \begin{pmatrix} -y \\ y \\ -y \end{pmatrix} \middle| \, y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\rangle \Rightarrow$$

f non è iniettiva perché $\mathrm{Ker} f \neq \{\vec{0}\}$

$$\boxed{B_{\mathrm{Ker} f} = \left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}}$$

$$\dim \mathbb{R}^3 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f \quad \Rightarrow \quad \boxed{\dim \operatorname{Im} f = 2}$$

$$3 = 1 + 2$$

$$\operatorname{Im} f = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 1 \\ -1 \\ -3 \end{pmatrix}} \right\rangle$$

$$\begin{pmatrix} -1 \\ 0 \\ -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -3 \end{pmatrix}$$

$$A \cdot \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -3 \end{pmatrix}$$

$$-v_1 + v_2 = v_3$$

$$-v_1 + v_2 - v_3 = \vec{0}$$

$$\boxed{B_{\operatorname{Im} f} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}}$$

b) $\operatorname{rg} A = \dim \operatorname{Im} f = 2$

Calcoliamo $f^{-1}\left\{ \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \right\}$ $\qquad \left(A \,\middle|\, \begin{matrix} 2 \\ 2 \\ 0 \\ -2 \end{matrix} \right)$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -3 \end{pmatrix}$$

Risolvere il sistema lineare con matrice completa

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & -1 & 0 \\ 2 & -1 & -3 & -2 \end{array} \right)$$

$$\begin{array}{l} \\ \\ 3^\circ - 1^\circ \\ 4^\circ - 2 \cdot 1^\circ \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & -3 & -3 & -6 \end{array} \right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{cases} x + y = 2 \\ y + z = 2 \end{cases} \qquad \begin{cases} \textcircled{x} = 2 - y \\ \textcircled{z} = 2 - y \end{cases} \qquad \begin{pmatrix} \textcircled{x} \\ y \\ \textcircled{z} \end{pmatrix}$$

$$\boxed{f^{-1}\left\{ \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \right\}} = \left\{ \begin{pmatrix} 2-y \\ y \\ 2-y \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\} = \boxed{\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\rangle}$$

$$\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -y \\ y \\ -y \end{pmatrix} \qquad y = 0$$

$$\boxed{f^{-1}\left\{ \begin{pmatrix} 2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \right\} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \operatorname{Ker} f = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\rangle}$$

iii) $B = \begin{pmatrix} 2 & -4 & 5 \\ -2 & 4 & 1 \\ 0 & 0 & 12 \end{pmatrix}$

$\underline{p_B(x)}$

$\underline{\text{autovalori di } B}$

autospazi

$\underline{B \text{ è diagonalizzabile?}}$

$$p_B(x) = \det(B - x\mathbb{I}_3) = \det \begin{pmatrix} 2-x & -4 & 5 \\ -2 & 4-x & 1 \\ 0 & 0 & 12-x \end{pmatrix} =$$

$$= (12-x)\,((2-x)(4-x) - 8) = (12-x)\,(\cancel{8} - 6x + x^2 - 8) = (12-x)\,x(x-6)$$

$$= (12-x) \,\Big|\, (2-x)(4-x) - 8) = (12-x)\underset{x^2-6x}{(\cancel{8}-6x+x^2\cancel{-8})} = (12-x)\,x(x-6)$$

$$\boxed{P_B(x) = (12-x)\,x\,(x-6)} \qquad \boxed{\text{Autovalori} \quad d_1 = 12,\ d_2 = 0,\ d_3 = 6}$$

$$\boxed{\begin{array}{l} m_a(12) = 1 = m_g(12) \\ m_a(0) = 1 = m_g(0) \\ m_a(6) = 1 = m_g(6) \end{array}}$$

perché $1 \le m_g(d) \le m_a(d)$

$\Longrightarrow$ B è diagonalizzabile $\quad B \sim D = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

per il teorema di diagonalizzabilità.

Calcoliamo gli autospazi:

$$V_{12} = \operatorname{Ker}(B - 12\,\mathbb{I}_3) = \operatorname{Ker}\begin{pmatrix} -10 & -4 & 5 \\ -2 & -8 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} -10x - 4y + 5z = 0 \\ -2x - 8y + z = 0 \end{cases} \qquad \begin{cases} -1\cancel{0x} - 4y + \cancel{10x} + 40y = 0 \\ z = 2x + 8y \end{cases}$$

$$\begin{cases} y = 0 \\ z = 2x \end{cases} \qquad \begin{pmatrix} x \\ \textcircled{y} \\ \textcircled{z} \end{pmatrix} \qquad \begin{pmatrix} x \\ 0 \\ 2x \end{pmatrix} \qquad \boxed{V_{12} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle}$$

$$V_0 = \operatorname{Ker}(B - 0\,\mathbb{I}_3) = \operatorname{Ker}\begin{pmatrix} 2 & -4 & 5 \\ -2 & 4 & 1 \\ 0 & 0 & 12 \end{pmatrix} \qquad \begin{cases} 2x - 4y + \cancel{5z} = 0 \\ -2x + 4y + \cancel{z} = 0 \\ 12\cancel{z} = 0 \end{cases}$$

$$\begin{cases} x = 2y \\ z = 0 \end{cases} \qquad \begin{matrix} \textcircled{x} \\ y \\ \textcircled{z} \end{matrix} \qquad \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix} \qquad \boxed{V_0 = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle}$$

$$B = \begin{pmatrix} 2 & -4 & 5 \\ -2 & 4 & 1 \\ 0 & 0 & 12 \end{pmatrix}$$

$$V_6 = \operatorname{Ker}(B - 6\,\mathbb{I}_3) = \operatorname{Ker}\begin{pmatrix} -4 & -4 & 5 \\ -2 & -2 & 1 \\ 0 & 0 & 6 \end{pmatrix} \qquad \begin{cases} -4x - 4y + \cancel{5z} = 0 \\ -2x - 2y + \cancel{z} = 0 \\ 6z = 0 \end{cases} \qquad \begin{cases} \textcircled{x} = -y \\ \textcircled{z} = 0 \end{cases} \qquad \begin{pmatrix} \textcircled{x} \\ y \\ \textcircled{z} \end{pmatrix}$$

$$\begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} \qquad \boxed{V_6 = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle}$$

$$H^{-1} B H = D \qquad H = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix} \qquad \det H = 2 \cdot 3 \neq 0$$

$$D = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad 12 \quad 0 \quad 6$$

$$V_{12} = \langle w_1 \rangle \qquad V_0 = \langle w_2 \rangle \qquad V_6 = \langle w_3 \rangle \qquad H = (w_1\ w_2\ w_3)$$

$$D = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

iv) Dimostriamo che $C=\begin{pmatrix} 0 & 1 & -4 \\ 0 & 12 & 21 \\ 0 & 0 & 6 \end{pmatrix}$ è simile a B.

Abbiamo dimostrato in iii) che $B \sim D=\begin{pmatrix} 12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$H^{-1}BH=D$ con $H \in GL_3(\mathbb{R})$

Dimostriamo che $C \sim D$ $\quad P_C(x) = \det\begin{pmatrix} -x & 1 & -4 \\ 0 & 12-x & 21 \\ 0 & 0 & 6-x \end{pmatrix} = -\underset{0}{x}(\underset{12}{12-x})(\underset{6}{6-x})$

$\Rightarrow$ $\boxed{\begin{array}{lll} C \text{ ha autovalori} & d_1 = 12 & m_a(12)=1=m_g(12) \\ & d_2=0 & m_a(0)=1=m_g(0) \\ & d_3=6 & m_a(6)=1=m_g(6) \end{array}}$ perciò C è

diagonalizzabile e $\boxed{C \sim D}$ cioè $\exists\, H' \in GL_3(\mathbb{R})$:

$(H')^{-1}CH'=D$ $\quad H'=(z_1\ z_2\ z_3)$ $\quad \langle z_1\rangle = \mathrm{Ker}(C-12I_3)$

$\langle z_2\rangle = \mathrm{Ker}(C-0I_3)$

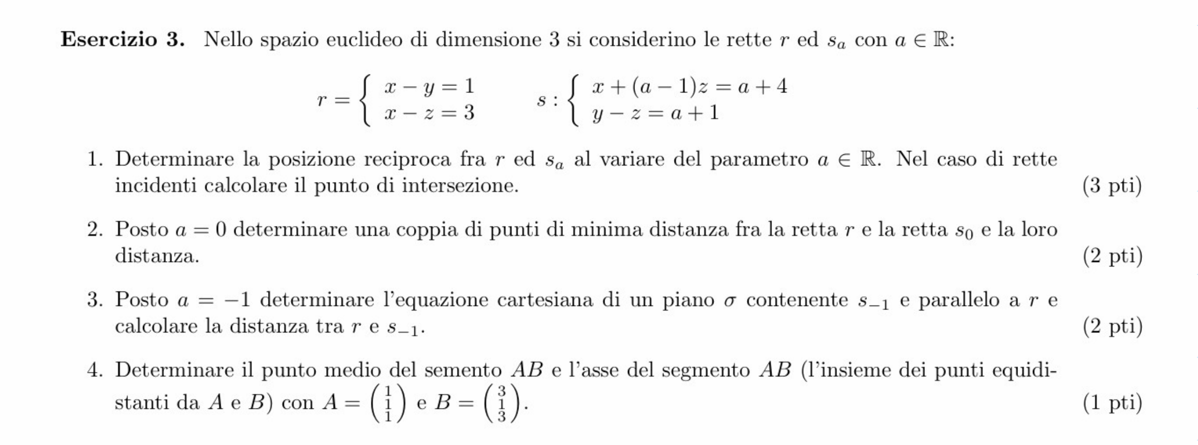
$\langle z_3\rangle = \mathrm{Ker}(C-6I_3)$

Essendo $\begin{array}{l} B \sim D \\ C \sim D \end{array}$ $\rightarrow$ $\underline{B \sim C}$ cioè $\exists\, K \in GL_3(\mathbb{R})$: $\underline{K^{-1}BK=C}$

$H^{-1}BH=D$ $\quad \boxed{H^{-1}BH=(H')^{-1}CH'}$ $\quad H'H^{-1}BH=\cancel{H'(H')^{-1}}CH'$

$(H')^{-1}CH'=D$ $\quad H'H^{-1}BH(H')^{-1}=C\cancel{H'(H')^{-1}}=C$

$H'H^{-1}B\,H(H')^{-1}=C$ $\quad \rightarrow \quad K=H(H')^{-1}$



Esercizio 3. Nello spazio euclideo di dimensione 3 si considerino le rette r ed s_a con $a \in \mathbb{R}$:

$$r = \begin{cases} x-y=1 \\ x-z=3 \end{cases} \qquad s: \begin{cases} x+(a-1)z=a+4 \\ y-z=a+1 \end{cases}$$

1. Determinare la posizione reciproca fra r ed s_a al variare del parametro $a \in \mathbb{R}$. Nel caso di rette incidenti calcolare il punto di intersezione. (3 pti)
2. Posto $a=0$ determinare una coppia di punti di minima distanza fra la retta r e la retta s_0 e la loro distanza. (2 pti)
3. Posto $a=-1$ determinare l'equazione cartesiana di un piano σ contenente s_{-1} e parallelo a r e calcolare la distanza tra r e s_{-1}. (2 pti)
4. Determinare il punto medio del semento AB e l'asse del segmento AB (l'insieme dei punti equidistanti da A e B) con $A=\begin{pmatrix}1\\1\\1\end{pmatrix}$ e $B=\begin{pmatrix}3\\1\\3\end{pmatrix}$. (1 pti)

Svolgimento:

1) $r \cap s_a$ $\quad \begin{cases} x-y=1 \\ x-z=3 \\ x+(a-1)z=a+4 \\ y-z=a+1 \end{cases}$ $\quad (A|b)=\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 3 \\ 1 & 0 & a-1 & a+4 \\ 0 & 1 & -1 & a+1 \end{array}\right)$

Riduciamo con Gauss $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & a-1 & a+3 \\ 0 & 1 & -1 & a+1 \end{array}\right)$ $\left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & \boxed{a} & a+1 \\ 0 & 0 & 0 & \boxed{a-1} \end{array}\right)$

$$\boxed{\text{Se } a \in \mathbb{R} \setminus \{0,1\} \quad rg(A)=3 \neq rg(A|b)=4 \Rightarrow \quad r \text{ e } s_a \text{ sono sghembe}}$$

Se $a=0$ $(A|b) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$ $\left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{array}\right)$

$rg A = 2 \neq rg(A|b) = 3 \Rightarrow$ $\boxed{r \text{ e } s_0 \text{ sono parallele disgiunte}}$

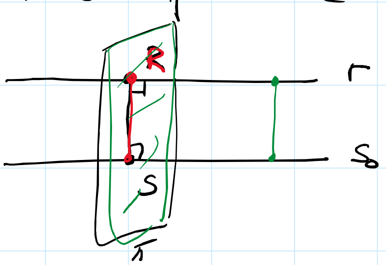
Se $a=1$ $(A|b) = \left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $rg(A) = 3 = rg(A|b)$

$$\begin{cases} x-y=1 \\ y-z=2 \\ z=2 \end{cases} \quad \begin{cases} x=y+1=4+1=5 \\ y=z+2=4 \\ z=2 \end{cases}$$

$$\boxed{r \text{ e } s_1 \text{ sono incidenti nel punto } P = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}}$$

b) Punti di minima distanza fra r e s_0 e $d(r, s_0)$.

$r \parallel s_0$ $\quad r \cap s_0 = \emptyset$



$d(r,s) = d(R,S)$

(R,S) coppia di minima distanza

$r: \begin{cases} x-y=1 \\ x-z=3 \end{cases} \quad \begin{cases} y=x-1 \\ z=x-3 \end{cases} \quad \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} = R \qquad S = \pi \cap s_0$

$s_0: \begin{cases} x-z=4 \\ y-z=1 \end{cases}$ determiniamo π piano $\perp$ ad r (e quindi anche a s_0) e passante per $R = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$

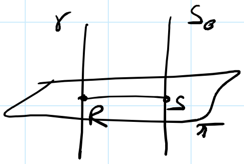
$V_r: \begin{cases} x-y=0 \\ x-z=0 \end{cases} \quad \begin{cases} y=x \\ z=x \end{cases} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{pmatrix} x \\ x \\ x \end{pmatrix} \quad V_r = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle = V_s \qquad 1x+1y+1z=K$

$x+y+z=K \quad \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$

$0-1-3=K$

$\pi: x+y+z=-4$

$\pi \cap s_0 \begin{cases} x+y+z=-4 \\ x-z=4 \\ y-z=1 \end{cases} \quad \begin{cases} 4+\underline{z}+1+\underline{z}+\underline{z}=-4 \\ x=4+z \\ y=1+z \end{cases}$



$\begin{cases} 3z=-9 \\ x=4+z \\ y=1+z \end{cases}$ $\begin{cases} z=-3 \\ x=1 \\ y=1-3=-2 \end{cases}$ $S=\begin{pmatrix}1\\-2\\-3\end{pmatrix}$

$$\boxed{\begin{array}{l} CMD\ (R,S)=\left(\begin{pmatrix}0\\-1\\-3\end{pmatrix},\begin{pmatrix}1\\-2\\-3\end{pmatrix}\right) \\ d(r,s_0)=\sqrt{2} \end{array}}$$

$$d(r,s_0)=d(R,S)=\|S-R\|=\left\|\begin{pmatrix}1\\-2\\-3\end{pmatrix}-\begin{pmatrix}0\\-1\\-3\end{pmatrix}\right\|=\left\|\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\|=\sqrt{2}$$

3) Posto $a=-1$ $s_{-1}:\begin{cases} x-2z=3 \\ y-z=0 \end{cases}$ $r:\begin{cases} x-y=1 \\ x-z=3 \end{cases}$ $V_r:\begin{cases} x-y=0 \\ x-z=0 \end{cases}$

r e s_{-1} sono sghembe per il punto 1).

Cerchiamo σ: $s_{-1}\subseteq\sigma$
$\sigma /\!/ r$

$s_{-1}:\begin{cases} x-2z-3=0 \\ y-z=0 \end{cases}$

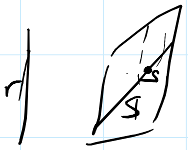
σ appartiene al fascio di sostegno s_{-1} $\sigma: a(x-2z-3)+b(y-z)=0$

$\sigma /\!/ r \iff V_r\subseteq V_\sigma$ $V_r=\left\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rangle$

$V_\sigma: a(x-2z)+b(y-z)=0$

$V_r=\left\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rangle\subseteq V_\sigma \iff \begin{pmatrix}1\\1\\1\end{pmatrix}\in V_\sigma$ $a(1-2)+b(\cancel{1-1})=0$ $-a=0$ $\begin{array}{l}a=0\\b=1\end{array}$

$\boxed{\sigma: y-z=0}$



$\sigma: ax+by+cz=d$

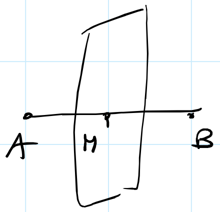
$\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot v_r=0$ $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot v_s=0$

$d(r,s)=d(r,\sigma)=d(R,\sigma)$

$r/\!/\sigma$ $R\in r$ $R=\begin{pmatrix}0\\-1\\-3\end{pmatrix}$ $\sigma: y-z=0$

$$d(r,s)=d(R,\sigma)=\frac{|-1+3|}{\sqrt{1+1}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

d) $A=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $B=\begin{pmatrix}3\\1\\3\end{pmatrix}$



$M=\frac{A+B}{2}$

Asse: $M+\langle B-A\rangle^{\perp}$

$M=\frac{A+B}{2}=\begin{pmatrix}2\\1\\2\end{pmatrix}$

$B-A=\begin{pmatrix}2\\0\\2\end{pmatrix}$ $\langle B-A\rangle=\left\langle\begin{pmatrix}2\\0\\2\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}1\\0\\1\end{pmatrix}\right\rangle$

$x+z=k$ passante per M

$2+2=k$

$$\boxed{\text{Asse: } x+z=4 \quad M=\begin{pmatrix}2\\1\\2\end{pmatrix}}$$