COMPUTABILITY (23/12/2025)

Computability
Jan 19 2022

definitions
proofs
small variations

Exercise 1

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if A is not recursive and $A \leqslant_m B$ then B is not recursive.

c. Show that if A is recursive then $A \leqslant_m \{1\}$.

constructions of PR / R
diagonalisation
smn

Exercise 2

Is there a non-computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = f(x+1)$ on infinitely many inputs x, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = f(x+1)\}$ is infinite? Provide an example or show that such a function cannot exist.

classify sets (recursive r.e.), saturatedness

Exercise 3

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is quasi-total if it is undefined on a finite number of inputs, i.e., $\overline{dom(f)}$ is finite. Classify the set $A = \{x \in \mathbb{N} \mid \varphi_x \text{ quasi-total}\}$ from the point of view of recursiveness, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 4

Classify the set $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ from the point of view of recursiveness, i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Note: Each exercise contributes with the same number of points (8) to the final grade.

Exercise 1

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if A is not recursive and $A \leqslant_m B$ then B is not recursive.

c. Show that if A is recursive then $A \leqslant_m \{1\}$.

(a) Given $A, B \subseteq \mathbb{N}$ we define $A \leq_m B$ if there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ s.t.

$\forall x \quad x \in A$ iff $f(x) \in B$

(b) We prove the counternominal: if B is recursive and $A \leq_m B$ then A is recursive

assume $A \leq_m B$, i.e. there is a total computable function $f: \mathbb{N} \to \mathbb{N}$

s.t. $\forall x \quad x \in A$ iff $f(x) \in B \qquad (*)$

Since B is recursive, the characteristic function

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

is computable

Then

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \overset{*}{\Longleftrightarrow} f(x) \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } f(x) \in B \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi_B(f(x))$$

is computable, since χ_B, f are computable.

Thus A is recursive, as desired.

(c) if A is recursive then $A \leq_m \{1\}$

Let $A \subseteq \mathbb{N}$ be recursive, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \qquad \text{computable}$$

Note that χ_A is total computable

$$x \in A \quad \text{iff} \quad \chi_A(x) = 1 \quad \text{iff} \quad \chi_A(x) \in \{1\}$$

hence χ_A reduces A to $\{1\}$.

ADDITIONAL QUESTION : Is it true that

if A recursive then $A \leq_m \{K\}$ $\quad K \in \mathbb{N}$ fixed ?

yes: $f(x) = \chi_A(x) \cdot K$ is the reduction function

ADDITIONAL QUESTION : Is the converse of point (c) true ?

If $A \leq_m \{1\}$ then A is recursive ?

YES: since $\{1\}$ is recursive (because it is finite)

DIRECT PROOF : let $f : \mathbb{N} \to \mathbb{N}$ be the reduction function for $A \leq_m \{1\}$ i.e. f total computable s.t.

$$\forall x \qquad x \in A \quad \text{iff} \quad f(x) \in \{1\} \quad \text{iff} \quad f(x) = 1$$

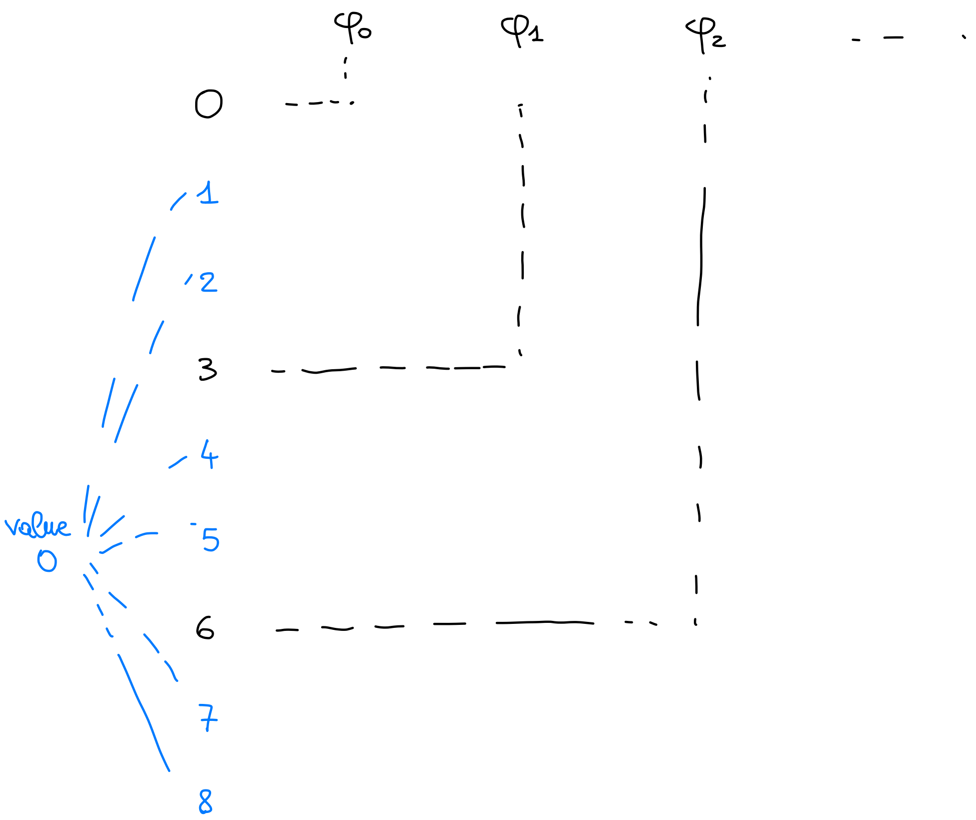
hence

$$\chi_A(x) = \overline{sg}(|f(x) - 1|) \qquad \text{computable.}$$

Exercise 2

Is there a non-computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = f(x+1)$ on infinitely many inputs x, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = f(x+1)\}$ is infinite? Provide an example or show that such a function cannot exist.

<u>solution 1</u>



$$g(x) = \begin{cases} \varphi_y(x) + 1 & \text{if } x = 3y \text{ for some } y \text{ and } \varphi_y(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

(if $x = 3y$ " " " " $\varphi_y(x)\uparrow$
or $x = 3y+1$ or $x = 3y+2$ for some y)

note that g is

→ total

→ not computable, since it differs from all total computable functions

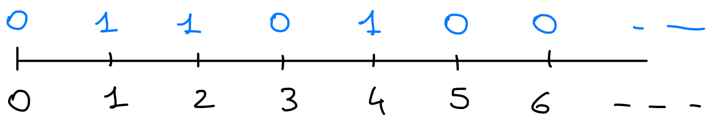
$\forall y$ if φ_y is total then $g(3y) = \varphi_y(3y)+1 \neq \varphi_y(3y)$

$\rightarrow \quad \{ x \mid f(x) = f(x+1) \} \supseteq \{ 3y+1 \mid y \in \mathbb{N} \}$

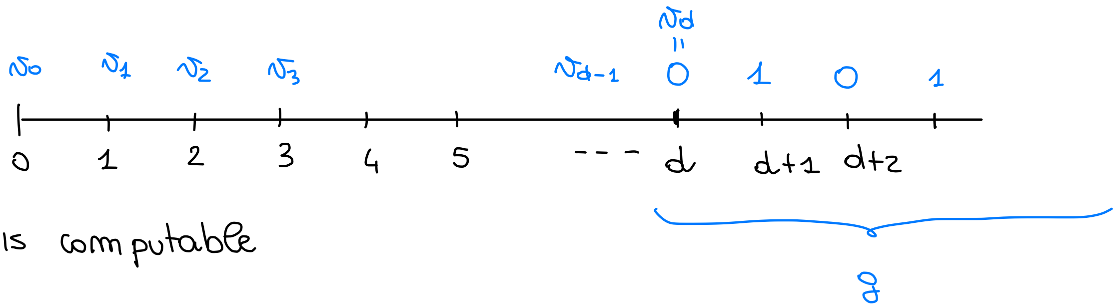
infinite.

solution 2

Consider $\chi_K : \mathbb{N} \to \mathbb{N}$



OBSERVATION: let $f : \mathbb{N} \to \mathbb{N}$ be a total function with $cod(f) \subseteq \{0,1\}$ and assume that there is $d \in \mathbb{N}$ s.t. $\forall x \geq d \quad f(x) \neq f(x+1)$



Then f is computable

In fact, let

$$f(x) = v_x \qquad \text{for} \quad x \leq d \quad ,$$

and assume (without loss of generality) $v_d = 0$

Define $g : \mathbb{N} \to \mathbb{N}$

$$\begin{cases} g(0) = 0 \\ g(x+1) = \overline{sg}(g(x)) \end{cases} \qquad \text{computable}$$

Then

$$f(x) = \prod_{i=0}^{d-1} \overline{sg}(|x-i|) \cdot v_i + g(x \dot{-} d)$$

- if $x < d$ then $x = i \quad 0 \leq i \leq d-1$

$$f(x) = v_i + \underbrace{g(\underbrace{i \dot{-} d}_{0})}_{0} = v_i$$

- if $x \geq d$

$$f(x) = 0 + g(x \dot{-} d)$$

Then f is computable by composition.

Hence the function g required by the exercise can be $g = \chi_K$

- χ_K total
- χ_K not computable
- $\{x \mid \chi_K(x) = \chi_K(x+1)\}$ is infinite

  (otherwise by the observation above, χ_K would be computable).

Exercise 3

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is quasi-total if it is undefined on a finite number of inputs, i.e., $\overline{dom(f)}$ is finite. Classify the set $A = \{x \in \mathbb{N} \mid \varphi_x \text{ quasi-total}\}$ from the point of view of recursiveness, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

conjecture : $A, \bar{A}$ not r.e. (hence not recursive)

A is saturated

$A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

$\mathcal{A} = \{ f \mid f \text{ quasi total} \} = \{ f \mid \overline{dom(f)} \text{ finite} \}$

* A is not r.e.

observe that $id \in \mathcal{A}$ (since $\overline{dom(id)} = \overline{\mathbb{N}} = \emptyset$ finite)

$\forall \vartheta \subseteq id$ ϑ finite $\overline{dom(\vartheta)} = \mathbb{N} \setminus dom(\vartheta)$ infinite

hence $\vartheta \notin \mathcal{A}$

Thus by Rice-Shapiro , A not r.e.

* $\bar{A}$ is not r.e. ($\bar{\mathcal{A}} = \{ f \mid \overline{dom(f)} \text{ infinite} \}$)

note that $id \notin \bar{\mathcal{A}}$

$\vartheta = \emptyset \subseteq id$ $\vartheta \in \bar{\mathcal{A}}$ ($\overline{dom(\emptyset)} = \bar{\emptyset} = \mathbb{N}$ infinite)

then by Rice-Shapiro , $\bar{A}$ not r.e.

Exercise 4

$\exists z\ \underbrace{\varphi_x(z)}_{y} > 2x$

Classify the set $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ from the point of view of recursiveness, i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

conjecture : B r.e. , not recursive

$\hookrightarrow$ $\bar{B}$ not r.e. (hence not recursive)

* <u>B is r.e.</u>

in fact

$y = 2x + 1 + h$

$$sc_B(x) = \mathbb{1}\left(\mu(z, y, t).\ \left(S(x, z, y, t) \wedge \overbrace{y > 2x}\right)\right)$$

$$= \mathbb{1}\left(\mu(z, h, t).\ S(x, z, 2x+1+h, t)\right)$$

$$= \mathbb{1}\left(\mu w.\ S(x, (w)_1, 2x+1+(w)_2, (w)_3)\right)$$

$$= \mathbb{1}\left(\mu w.\ |\chi_S(x, (w)_1, 2x+1+(w)_2, (w)_3) - 1|\right)$$

computable

* <u>B not recursive</u>

We show that K (not recursive) reduces to B $\qquad K \le_m B$

We need a reduction function $s: \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x$

$$x \in K \quad \text{iff} \quad s(x) \in B$$

$\circlearrowleft\ \exists z\ \varphi_{s(x)}(z) > 2s(x)$

define

$$g(x, y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases} = y \cdot sc_K(x)$$

computable

By smn theorem, there is $s: \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$

We claim that s is the reduction function for $K \leq_m B$

* if $x \in K$ then

$$\varphi_{s(x)}(y) = y \quad \forall y$$

and thus if $y = 2s(x)+1 > 2s(x)$ then $\varphi_{s(x)}(y) = y > 2s(x)$

Hence $s(x) \in B$

* If $x \notin K$ then

$$\varphi_{s(x)}(y) \uparrow \quad \forall y$$

and thus there is no $y \in \underset{\emptyset}{\underset{\|}{E_{s(x)}}}$ s.t. $y > 2s(x)$.

Thus $s(x) \notin B$

Hence $K \leq_m B$ and thus B is not recursive.

Summing up, B r.e., not recursive

hence $\bar{B}$ not r.e. (otherwise if $\bar{B}$ r.e., since also B r.e.

we would have B recursive)

and thus B not recursive. □

ADDITIONAL QUESTION : Is B saturated ?

$$B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$$

NO : We show that there are $e, e' \in \mathbb{N}$ s.t. $\varphi_e = \varphi_{e'}$ but

$e \in B$, $e' \notin B$

We show that there is $e \in \mathbb{N}$

$$\varphi_e(y) = 2e+1$$

Define

$$g(x,y) = 2x+1 \qquad \text{computable}$$

hence by smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x, y \qquad \varphi_{s(x)}(y) = g(x,y) = 2x+1$

Since s is total computable by the 2nd recursion theorem there is $e \in \mathbb{N}$

s.t. $\varphi_e = \varphi_{s(e)}$

Thus $\qquad \varphi_e(y) = \varphi_{s(e)}(y) = g(e,y) = 2e+1$

Observe :

- $e \in B$ since $E_e = \{\underset{\vee \atop 2e}{2e+1}\}$

- if $e' > e$ s.t. $\varphi_{e'} = \varphi_e$ (it exists for sure since there are infinitely many indices for a computable function)

  then

  $e' \notin B$ since $E_{e'} = E_e = \{2e+1\}$

  and $2e+1 < 2e+2 = 2(\underset{\wedge \atop e'}{e+1}) \leq 2e'$

Therefore B is not saturated. $\square$